Contributions to non local evolution equations in space-time

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à mes parents

à ma fiancée Ghada

à mon frère Rabah

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Chapitre 1

Introduction

Dans cette thèse, nous nous intéressons à l'étude de quatre équations d'évolution non-locales (cf. (1.2.1), (1.3.1), (1.4.1) et (1.5.1) ci-dessous). Les solutions de ces quatre équations peuvent exploser en temps fini.

Dans la théorie des équations d'évolution non-linéaires, une solution est qualifiée de globale si elle est définie pour tout temps positif. Au contraire, si une solution existe seulement sur un intervalle de temps $[0, T)$ borné, elle est dite locale. Dans ce dernier cas et quand le temps maximal d'existence est relié à une alternative d'explosion, on dit aussi que la solution explose en temps fini. Cependant, pour donner un sens à la notion d'explosion en temps fini, il faut bien préciser l'espace dans lequel on travaille et avec quelle norme on "mesure" la solution.

Dans le deuxième chapitre, nous considérons l'équation de Schrödinger non-linéaire avec une puissance fractionnaire du laplacien, et nous obtenons que la solution explose en temps fini $T_{\max} > 0$ pour toute condition initiale positive et non-triviale dans le cas d'exposant sous-critique.

Dans le troisième chapitre, nous étudions une équation des ondes amorties avec un potentiel d'espace-temps et un terme non-linéaire et non-local en temps. Nous obtenons un résultat d'existence locale d'une solution dans l'espace d'énergie sous des conditions restrictives sur les données initiales, la dimension de l'espace et de la croissance du terme non-linéaire. De plus, nous obtenons que la solution explose en temps fini pour toute condition initiale de moyenne strictement positive.

Le quatrième chapitre est consacré à l'étude du problème de Cauchy pour l'équation d'évolution p-Laplacien avec une mémoire non linéaire. On étudie l'existence locale d'une solution ainsi qu'un résultat de non-existence de solution globale.

Finalement, dans le cinquième chapitre, nous étudions l'intervalle maximal d'existence des solutions du problème d'évolution dans un milieu poreux avec un terme non-linéaire non-local en temps. De plus, nous obtenous que la solution n'existe pas globalement sous certaines conditions sur les données initiales.

Pour mettre en évidence les points importants et afin d'éviter qu'ils ne soient cachés dans les détails techniques, nous donnons dans cette introduction générale, des énoncés simplifiés des résultats. Les énoncés complets se trouvent dans les différents chapitres présentés ci-dessous.

1.1 Préliminaires

Soit $AC[0,T]$ l'espace de toutes les fonctions absoluement continues sur $[0,T]$ avec $0 < T < \infty$. Alors pour $f \in AC[0,T]$, on définit les dérivées fractionnaires de Riemann-Liouville à droite et à gauche de f d'ordre $\alpha \in (0,1)$ par

$$D^{\alpha}_{0|t}f(t) := \partial_t J^{1-\alpha}_{0|t}f(t) \quad \text{et} \quad D^{\alpha}_{t|T}f(t) := -\frac{1}{\Gamma(1-\alpha)}\partial_t \int_t^T (s-t)^{-\alpha} f(s)\, ds, \quad t \in [0,T], \quad (1.1.1)$$

où $\partial_t = \dfrac{d}{dt}$ et

$$J^{\alpha}_{0|t}g(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds \quad (1.1.2)$$

est l'intégrale fractionnaire de Riemann-Liouville pour toute fonction $g \in L^q(0,T)$ et $1 \leq q \leq \infty$. (Voir [37]).

De plus, pour toutes $f, g \in C([0,T])$ telles que $D^{\alpha}_{0|t}f(t), D^{\alpha}_{t|T}g(t)$ existent et sont continues, pour tout $t \in [0,T]$, $0 < \alpha < 1$, on a la formule d'intégration par partie suivante (Voir [64])

$$\int_0^T \left(D^{\alpha}_{0|t}f\right)(t)g(t)\, dt \;=\; \int_0^T f(t)\left(D^{\alpha}_{t|T}g\right)(t)\, dt. \quad (1.1.3)$$

Notons que pour toute $f \in AC^{n+1}[0,T]$ et tout entier $n \geq 0$, on a (voir (2.2.30) dans [37])

$$(-1)^n \partial_t^n . D^{\alpha}_{t|T}f = D^{n+\alpha}_{t|T}f, \quad (1.1.4)$$

où

$$AC^{n+1}[0,T] := \{f : [0,T] \to \mathbb{R} \text{ et } \partial_t^n f \in AC[0,T]\}$$

et ∂_t^n est la n-ième dérivée. De plus, pour tout $1 \leq q \leq \infty$, la formule suivante (voir [37, Lemma 2.4 p.74])

$$D^{\alpha}_{0|t}J^{\alpha}_{0|t} = Id_{L^q(0,T)} \quad (1.1.5)$$

est valable presque partout sur $[0,T]$.

Dans ce qui suit, on utilise les résultats suivants :

Si $w_1(t) = (1 - t/T)^{\sigma}_{+}$, $t \geq 0$, $T > 0$, $\sigma \gg 1$, alors

$$D^{\alpha}_{t|T}w_1(t) = CT^{-\sigma}(T-t)^{\sigma-\alpha}_{+}, \quad D^{\alpha+1}_{t|T}w_1(t) = CT^{-\sigma}(T-t)^{\sigma-\alpha-1}_{+}, \quad D^{\alpha+2}_{t|T}w_1(t) = CT^{-\sigma}(T-t)^{\sigma-\alpha-2}_{+}, \quad (1.1.6)$$

pour toute $\alpha \in (0,1)$; donc

$$\left(D^{\alpha}_{t|T}w_1\right)(T) = 0, \quad \left(D^{\alpha}_{t|T}w_1\right)(0) = C\, T^{-\alpha}, \quad \left(D^{\alpha+1}_{t|T}w_1\right)(T) = 0 \quad \text{et} \quad \left(D^{\alpha+1}_{t|T}w_1\right)(0) = C\, T^{-\alpha-1}. \quad (1.1.7)$$

Pour la preuve de ces résultats, voir [21, Preliminaries].

1.2 Explosion en temps fini pour une équation de Schrödinger non-linéaire avec un laplacien fractionnaire

Dans le deuxième chapitre, nous nous intéressons à l'équation de Schrödinger non-linéaire avec un laplacien fractionnaire :

$$\begin{cases} i\partial_t u = \Lambda^\alpha u + \lambda |u|^p & (t,x) \in [0,T) \times \mathbb{R}^N, \\ \\ u(x,0) = f(x) & x \in \mathbb{R}^N, \end{cases} \tag{1.2.1}$$

où le laplacien fractionnaire $\Lambda^\alpha = (-\Delta)^{\alpha/2}$ avec $0 < \alpha < 2$ est un opérateur pseudo-différentiel défini par la transformée de Fourier : $\widehat{\Lambda^\alpha u}(\xi) = |\xi|^\alpha \widehat{u}(\xi)$. De plus, nous supposons que $T > 0$, $1 < p \leq 1 + \frac{\alpha}{N}$, $u = u(x,t)$ est une fonction à valeur complexe, $\lambda \in \mathbb{C} \backslash \{0\}$ et $f = f(x) \in H^{\frac{\alpha}{2}}(\mathbb{R}^N)$ est une fonction donnée à valeur complexe.
En fait, le laplacien fractionnaire est un cas particulier de l'opérateur de Lévy $\mathcal{L}$ qui est bien un opérateur pseudo-differentiel défini par $\widehat{\mathcal{L}v} = a(\xi)\widehat{v}(\xi)$. Comme la fonction $e^{-ta(\xi)}$ est bien définie et positive, alors le symbole $a(\xi)$ peut se représenter, comme dans [8], par la formule de Lévy-Khintchine (cf. [6, Chapitre 1, Théorème 1] ou [34, Théorème B.2])

$$a(\xi) = ib\xi + q(\xi) + \int_{\mathbb{R}^N} (1 - e^{-i\eta\xi} - i\eta\xi \mathbb{1}_{\{|\eta|<1\}}(\eta))\Pi(\,d\eta).$$

Dans la littérature de la physique mathématique (voir [7, 19, 36]), les problèmes d'évolution non-linéaires avec un laplacien fractionnaire décrivent la diffusion anormale (the anomalous diffusion) ou ce qu'on appelle la diffusion $\alpha-$stable de Lévy ($\alpha-$stable Lévy diffusion).
Dans les dernières années, l'étude du calcul fractionnaire et des équations intégrodifférentielles fractionnaires a évoluée vue qu'elle s'applique à de nombreux domaines notamment la physique, (voir [37], [52],[53] et les références citées). Meltzler et Klafter ont discuté les développements récents dans la description de la diffusion anormale par l'approche de la dynamique fractionnaires dans [52] et [53] où de nombreuses équations aux dérivées partielles fractionnaires sont dérivées asymptotiquement du modèle de marche aléatoire de Lévy, qui représente une sorte de généralisation naturelle des modèles de marche Browniens. Inspiré par l'approche du chemin de Feynman à la mécanique quantique, Laskin a utilisé le chemin intégrale de Lévy pour obtenir une équation fractionnaire de Schrödinger. Ce qui généralise un résultat classique concernant l'intégrale du chemin sur des trajectoires browniens conduisant aux équations standards de Schrödinger (voir [43],[44]). Nous trouvons quelques papiers étudiants les équations fractionnaires de Schrödinger et leurs applications, (voir [26], [62] et les références citées).
Quand $\alpha = 2$, il est bien connu que (1.2.1) est bien posé localement dans $H^1(\mathbb{R}^N)$ si $1 < p < 1 + \frac{4}{(N-2)_+}$ (voir [11]). En outre, il est également connu que les solutions locales peuvent être prolongées en solutions globales en considérant des petites données lorsque p est plus grand que l'exposant de Strauss p_s qui est défini comme étant la racine positive de $Np^2 - (N+2)p - 2 = 0$ (voir [9]). Cependant, il n'y a pas de résultats d'existence globale pour $p \leq p_s$. En 2013, Ikeda et

Wakasugi [30] ont montré un résultat d'explosion pour la solution du problème (1.2.1) pour des petites données quand $1 < p \leq 1 + \frac{2}{N}$.
L'objectif principal de cette étude est de généraliser le résultat d'explosion de Ikeda et Wakasugi [30] pour des solutions douces des équations fractionnaires de Schrödinger dont la définition est donnée ci-dessous.

Définition 1.2.1 (Solution douce)
Soit $f \in H^{\frac{\alpha}{2}}(\mathbb{R}^N)$, $0 < \alpha \leq 2$, $p > 1$ et $T > 0$. Nous disons que $u \in C([0,T], H^{\frac{\alpha}{2}}(\mathbb{R}^N))$ est une solution douce du problème (1.2.1) si u est solution de l'équation intégrale

$$u(t) = S(t)f - i\lambda \int_0^t S(t-s)|u(s)|^p \, ds, \tag{1.2.2}$$

où le générateur infinitesimal A du C_0 groupe de l'opérateur unitaire $S(t)$ est défini par $Au = -i(-\Delta)^{\frac{\alpha}{2}}u$ pour $-\infty < t < \infty$, sur $L^2(\mathbb{R}^N)$.

L'existance locale est basée sur le théorème de point fixe de Banach, via la théorie des semi-groupes et le théorème de Stone concernant l'opérateur fractonnaire $A = -i(-\Delta)^{\frac{\alpha}{2}}$ qui est un générateur infinitésimal du groupe continu des opérateurs unitaires sur L^2 (voir [11]).
La méthode utilisée pour démontrer le résultat d'explosion est la méthode des fonctions tests développée par Zhang [66], puis par Pohozaev et Mitidieri [43].
Pour énoncer notre résultat principal, soit $\lambda = \lambda_1 + i\lambda_2$ et $f = f_1 + if_2$. Nous introduisons les hypothèses suivantes :

$$\begin{aligned} & f_1 \in L^1(\mathbb{R}^N), \quad \lambda_2 \int_{\mathbb{R}^N} f_1 \, dx > 0 \\ \textit{ou} & \\ & f_2 \in L^1(\mathbb{R}^N), \quad \lambda_1 \int_{\mathbb{R}^N} f_2 \, dx < 0 \end{aligned} \tag{1.2.3}$$

Notre résultat principal est le théorème suivant :

Théorème 1.2.2 (Explosion en temps fini)
Soit $\lambda \in \mathbb{C}\backslash\{0\}$, et $f \in H^{\frac{\alpha}{2}}(\mathbb{R}^N)$ satisfaisant à (1.2.3). Si $1 < p \leq 1 + \frac{\alpha}{N}$ alors la solution douce de (1.2.1) explose en temps fini.

1.3 Explosion en temps fini d'une solution de l'équation d'onde amortie avec un potentiel d'espace-temps et d'un terme mémoire non linéaire

Ce travail concerne le problème de Cauchy pour l'équation des ondes amorties semi-linéaire suivante

$$\begin{cases} u_{tt} - \Delta u + a(x)b(t)u_t = \displaystyle\int_0^t (t-s)^{-\gamma}|u(s)|^p\,ds, & t > 0, x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.3.1}$$

où u est une fonction à valeurs réelles, $n \geq 1$, $0 < \gamma < 1$ et $p > 1$. Le coefficient du terme d'amortissement s'écrit

$$a(x)b(t) := a_0(1+|x|^2)^{-\frac{\alpha}{2}}(1+t)^{-\beta},$$

avec $a_0 > 0$, $\alpha, \beta \geq 0$ et $\alpha + \beta < 1$. Dans ce travail, nous supposons que les données initiales sont dans l'espace d'énergie suivant

$$(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n). \tag{1.3.2}$$

Les notations suivantes $\|\cdot\|_q$ et $\|\cdot\|_{H^1}$ $(1 \leq q \leq \infty)$, représentent les normes des espaces $L^q(\mathbb{R}^n)$ et $H^1(\mathbb{R}^n)$, respectivement.
Le terme non local non linéaire peut être considéré comme une approximation (avec un changement de variable approprié) de la non-linéarité de l'équation des ondes amorties semi-linéaires suivantes

$$u_{tt} - \Delta u + a(x)b(t)u_t = |u(t)|^p,$$

car

$$\lim_{\gamma \to 1} s_+^{-\gamma} = \Gamma(1-\gamma)\delta(s)$$

existe au sens de distribution, où Γ est la fonction d'Euler.
Quand on a l'équation,

$$u_{tt} - \Delta u + u_t = |u(t)|^p, \tag{1.3.3}$$

Li et Zhou ont démontré dans [50] que si $n \geq 2$, $1 < p \leq 1 + 2/n$ et les données initiales sont positives en moyenne, alors la solution locale de (1.3.3) peut exploser en temps fini. Dans [65], Todorova et Yordanov ont développé une méthode d'énergie pondérée et ont déterminé que l'exposant critique de (1.3.3) est donné par

$$p_c = 1 + \frac{2}{n},$$

qui est bien connu comme l'exposant critique de Fujita pour l'équation de la chaleur $u_t - \Delta u = u^p$ (voir [24]). Plus précisément, ils ont démontré l'existence globale de solution pour des petites données initiales dans le cas $p > 1 + 2/n$. Cependant, pour le cas $1 < p < 1 + 2/n$, il y a explosion en

temps fini des solutions de (1.3.3).
Plus tard, Zhang [70] et indépendamment Kirane et Qafsaoui [39] ont montré que l'exposant critique $p_c = 1 + 2/n$ fait partie de la région d'explosion.
Todorova et Yordanov supposent que les données initiales ont un support compact et utilisent essentiellement cette propriété dans [65]. Cependant, Ikehata et Tanizawa suppriment cette hypothèse dans [32]. Ikehata et al. ont étudié dans [33] le cas où le coefficient dépend de l'espace

$$u_{tt} - \Delta u + a(x)u_t = |u|^p,$$

où

$$a(x) \equiv a_0(1 + |x|^2)^{-\alpha/2}, \quad |x| \to \infty, \qquad \text{symétrique radiale et } 0 \leq \alpha < 1.$$

Ils démontrent que l'exposant critique de (1.3.3) est donné par

$$p_c = 1 + \frac{2}{n - \alpha}$$

en utilisant une méthode de multiplicateurs raffinée. Leur méthode dépend aussi de la propriété de propagation à vitesse finie. Récemment, Nishihara [59] et Lin et al. [48] ont consideré une équation d'onde semi-linéaire avec un terme d'amortissemnet en temps

$$u_{tt} - \Delta u + b(t)u_t = |u|^p,$$

où

$$b(t) = b_0(1 + t)^{-\beta}, \quad \beta \in (-1, 1).$$

Ils ont montré que l'exposant critique est

$$p_c = 1 + \frac{2}{n}.$$

Cela montre que les coefficients dépendants du temps sous une forme spéciale dans le terme d'amortissement n'influencent pas l'exposant critique.
D'autre part, l'équation (1.3.1) peut être écrite sous la forme

$$u_{tt} - \Delta u + a(x)b(t)u_t = J^{\delta}_{0|t}\left(|u|^p\right)(t), \tag{1.3.4}$$

où $\delta = 1 - \gamma$, et $J^{\delta}_{0|t}$ est donné par (1.1.2). Nous présentons dans ce qui suit notre résultat principal. Nous annonçons d'abord le résultat d'existence locale de la solution du problème (1.3.1).

Définition 1.3.1 (Solution douce)
Soit $(u_0, u_1) \in H^1 \times L^2$. Une fonction u est dite solution douce de (1.3.1) avec données initiales $u(0) = u_0$ et $u_t(0) = u_1$ si

$$u \in C([0, T); H^1) \cap C^1([0, T); L^2)$$

et si u satisfait à l'équation intégrale suivante

$$u(t, x) = R(t, 0)(u_0, u_1) + \Gamma(\delta) \int_0^t S(t, s) J^{\delta}_{0|t}(|u|^p)\, ds \tag{1.3.5}$$

au sens $H^1(\mathbb{R}^n)$, *où* $\delta = 1-\gamma$, $S(t,s)g := R(t,s)(0,g)$ *pour toute fonction* $g \in H^1(\mathbb{R}^N)$ *et* $R(t,s)$ *est l'opérateur qui associe à chaque donnée initiale* $(u(s), u_t(s)) \in H^2 \times H^1$ *la solution* $u(t) \in H^2$ *pour* $t \geq s$. *La solution* u *du problème homogène associé à (1.3.1) est définie par* $u(t) = R(t,0)(u_0, u_1)$.

Proposition 1.3.2 (Existence et unicité de la solution douce)
Soit $\alpha \geq 0$, $\beta \in \mathbb{R}$, $1 < p \leq n/(n-2)$ *pour* $n \geq 3$, *et* $p \in (1,\infty)$ *pour* $n = 1,2$. *Sous les hypothèses (1.3.2) et* $\gamma \in (0,1)$, *le problème (1.3.1) admet une solution douce unique maximale* u, *c.à.d satisfait à l'équation intégrale (1.3.5), sachant que*

$$u \in C([0, T_{\max}), H^1(\mathbb{R}^n)) \cap C^1([0, T_{\max}), L^2(\mathbb{R}^n)),$$

où $0 < T_{\max} \leq \infty$. *De plus, si* $T_{\max} < \infty$, *alors* $\|u(t)\|_{H^1} + \|u_t(t)\|_2 \to \infty$ *quand* $t \to T_{\max}$.

Remarque 1.3.3
On dit que u *est une solution globale de (1.3.1) si* $T_{\max} = \infty$, *alors que dans le cas où* $T_{\max} < \infty$, *on dit que* u *explose en temps fini.*

Soit

$$p_c := 1 + \frac{3-\gamma}{(n-\alpha-1+\gamma)_+}.$$

comme

$$(p_c = n/(n-2)) \iff (\gamma = (n+2(\alpha-2))/n),$$

ceci implique, dans le cas où $(n+2(\alpha-2))/n \leq \gamma$, que $p_c \leq n/(n-2)$. Notons que

$$p_c \to 1 + \frac{2}{n-\alpha} \qquad \text{quand } \gamma \to 1.$$

Notre résultat principal est le suivant

Théorème 1.3.4 (Résultats d'explosion en temps fini)
Sous l'hypothèse $\alpha\beta = 0$, *on introduit*

$$p^* := 1 + \frac{3-\gamma}{(N-\alpha-1+\gamma)_+} \qquad \textit{et} \qquad \gamma^* := \frac{N+2(\alpha-2)}{N}.$$

Supposons que $(u_0, u_1) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ *et que*

$$\int_{\mathbb{R}^n} a(x) u_0 \, dx > 0, \qquad \int_{\mathbb{R}^n} u_1(x) \, dx > 0. \tag{1.3.6}$$

La solution douce du problème (1.3.1) explose en temps fini si
i)

$$\begin{cases} 1 < p \leq \dfrac{N}{N-2} \ \textit{et}\ \gamma \geq \gamma^* & N \geq 3, \\[2ex] p \in (1,\infty) & N = 1,2. \end{cases} \qquad \textit{et} \qquad p \leq p^*$$

ii) $N \geq 3$, $1 < p \leq N/(N-2)$ *et* $\gamma \leq \gamma^*$.

La méthode des fonctions tests (voir [21, 22, 23, 38, 54, 55, 70] et les références citées) est l'idée principale sur laquelle est basée la preuve du résultat d'explosion.

Remarque 1.3.5
On ne connait pas de résultat d'existence de solution globale du problème (1.3.1) quand $p > p_c$.

1.4 Sur le problème de Cauchy pour l'équation d'évolution avec le p-Laplacien et un terme mémoire non linéaire

Cette partie est consacrée à l'étude du problème suivant de Cauchy pour l'équation d'évolution

$$\begin{cases} u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \displaystyle\int_0^t (t-s)^{-\gamma}|u(s)|^{q-1}u(s)\,ds, & t>0, x\in\mathbb{R}^n, \\ u(0,x) = u_0(x), & x\in\mathbb{R}^n, \end{cases} \tag{1.4.1}$$

où $p > 2$, $q > 1$, $0 < \gamma < 1$ et $u_0 \in L^\infty(\mathbb{R}^n)$.
Le premier résultat sur l'explosion de la solution pour l'équation de la chaleur a été obtenu par Fujita dans [24]. Il a étudié le problème de Cauchy suivant

$$\begin{cases} u_t - \Delta u = u^p, & t>0, x\in\mathbb{R}^n, \\ u(0,x) = u_0(x), & x\in\mathbb{R}^n, \end{cases} \tag{1.4.2}$$

où $p > 1$. Il a obtienu les résultats suivants :
a) Si $p < 1 + 2/n$, alors chaque solution non-triviale de (1.4.2) explose en temps fini.
b) Si $p > 1 + 2/n$ et $u_0(x) \leq \delta e^{-|x|^2}$, $(0 < \delta \ll 1)$, alors (1.4.2) admet une solution globale.
Dans le cas $p = 1 + 2/n$, il est démontré par l'auteur de [27] pour $n = 1, 2$ et par les auteurs de [40] pour $n \geq 1$ que (1.4.2) n'a pas de solution globale $u(x,t)$ satisfaisant $\|u(\cdot\,,t)\|_\infty < \infty$ pour $t \geq 0$. L'auteur de [68] démontre que si $p = 1 + 2/n$, (1.4.2) n'a pas de solution globale $u(x,t)$ satisfaisant $\|u(\cdot\,,t)\|_q < \infty$ pour $t > 0$ et pour $q \in [1, +\infty)$. La valeur $p_c = 1 + 2/n$ est appelé l'exposant critique de (1.4.2). Il joue un rôle important dans le comportement de la solution de (1.4.2).

Dans les dernières années, il y a eu un certain nombre d'extensions des résultats de Fujita dans de nombreuses directions. Récemment, les auteurs de [10] ont étendu les résultats de Fujita pour l'équation de la chaleur avec un terme de mémoire non-linéaire

$$\begin{cases} u_t - \Delta u = \displaystyle\int_0^t (t-s)^{-\gamma}u(s)^p\,ds, & t>0, x\in\mathbb{R}^n, \\ u(0,x) = u_0(x), & x\in\mathbb{R}^n, \end{cases} \tag{1.4.3}$$

où $p > 1$, $0 < \gamma < 1$, et $u_0 \in C_0(\mathbb{R}^n)$. Dans ce cas, la valeur de l'exposant critique de Fujita est $p_c = \max\{\frac{1}{\gamma}, 1 + 2(2-\gamma)/(N-2+2\gamma)_+\}$. De plus, le terme non local non-linéaire peut être considéré comme une approximation du terme non-linéaire de l'équation de la chaleur semi-linéaire (1.4.3) puisque la limite

$$\lim_{\gamma\to 1} s_+^{-\gamma} = \Gamma(1-\gamma)\delta(s)$$

existe au sens des distributions, où Γ est la fonction gamma d'Euler.
Pour les équations d'évolution avec un p-Laplacien

$$\begin{cases} u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = u^q, & t>0, x\in\mathbb{R}^n, \\ u(0,x) = u_0(x), & x\in\mathbb{R}^n, \end{cases} \tag{1.4.4}$$

où $p>2$ et $q>1$, l'auteur de [71] a étudié le problème de Cauchy ; et il a obtenu l'exposant critique $q_c = p-1+p/n$ à condition que $p>2n/(n+1)$. En outre, l'auteur démontre que si $\max\{1,p-1\} < q < q_c$, alors le problème de Cauchy n'a pas de solution globale ; tandis que si $q>q_c$ et $u_0(x)$ est assez petite, alors le problème de Cauchy admet une solution globale positive. Les auteurs de [2, 49] discutent, respectivement, des équations paraboliques quasi-linéaires plus généraux et des équations paraboliques doublement singulières, et obtiennent des résultats similaires.

Notre objectif dans ce travail est d'étudier l'existence locale de solutions de l'équation (1.4.1), puis d'étudier la non-existence de la solution globale.

En utilisant l'intégrale de Riemann-Liouville, (1.4.1) s'écrit sous la forme

$$u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \Gamma(\delta) J^{\delta}_{0|t}\left(|u|^p\right)(t), \tag{1.4.5}$$

où $\delta = 1-\gamma$.
Tout d'abord, nous intoduisons la définition suivante

Définition 1.4.1 (Solution faible)
Une fonction $u(x,t)$ définie sur $S_T = \mathbb{R}^N \times (0,T]$ est dite solution faible de (1.4.3) si pour tout ensemble ouvert Ω avec un bord régulier $\partial\Omega$,

$$u \in C([0,T]:L^1(\Omega)) \cap L^p_{loc}(0,T:W^1_p(\Omega)) \cap L^\infty_{loc}(S_T),$$

et

$$\begin{aligned} \int_\Omega u(x,t)\phi(x,t)\,dx + \int_{t_0}^t \int_\Omega [-u\phi_t + |\nabla u|^{p-2}\nabla u\cdot\nabla\phi]\,dx\,d\tau \\ = \Gamma(\delta)\int_{t_0}^t\int_\Omega J^\delta_{0|\tau}(|u|^{q-1}u)\varphi\,dxd\tau + \int_\Omega u(x,t_0)\phi(x,t_0)\,dx \end{aligned} \tag{1.4.6}$$

pour tout $0\le t_0 < t \le T$ et $\phi\in C^1(\overline{\Omega}\times[0,T])$, $\phi(\cdot,T)=0$ au voisinage de $\partial\Omega$, où $\delta = 1-\gamma$. De plus,

$$\lim_{t\mapsto 0}\int_{B_r} |u(x,t)-u_0(x)|\,dx = 0, \quad \forall r>0. \tag{1.4.7}$$

Nous utilisons $\nu(a_1, a_2, \cdots, a_n)$ pour définir des constantes positives dépendants seulement de $a_1, a_2, \cdots, a_n$. Soit

$$\operatorname{sgn}_\eta s = \begin{cases} 1 & \text{si} \quad s > \eta, \\ \dfrac{s}{\eta} & \text{si} \quad -\eta \leq s \leq \eta, \\ -1 & \text{si} \quad s < -\eta. \end{cases}$$

Pour $f \in L_{loc}(\mathbb{R}^N)$, nous définissons

$$|||f|||_h = \sup_{x \in \mathbb{R}^N} \left(\frac{1}{|B_1(x)|} \int_{B_1(x)} |f(y)|^h \, dy \right)^{\frac{1}{h}}. \tag{1.4.8}$$

Théorème 1.4.2 (Existence locale)
Soit $|||u_0|||_h < \infty$ *où*

$$h = 1 \quad si \quad 1 \leq q < p - 1 + \frac{p}{N} \quad et \quad h > \left(\frac{N}{p}\right)(q - p + 1) \quad si \quad q \geq p - 1 + \frac{p}{N}. \tag{1.4.9}$$

Alors il existe une constante $\nu = \nu(N, p, q) \geq 1$ *et une constante positive* T_0 *définie par*

$$T_0 + T_0 |||u_0|||_h^{p-2} + T_0^{1 - \frac{N(q-p+1)}{ph}} |||u_0|||_h^{q-1} = \nu^{-1} \tag{1.4.10}$$

et une solution faible u *to (1.4.1) dans* S_T *satisfaisant, pour tout* $0 < t < T_0$,

$$|||u(\cdot, t)|||_h \leq \nu |||u_0|||_h, \tag{1.4.11}$$

$$|u(x, t)| \leq \nu t^{-\frac{N}{k_h} + \frac{\gamma - 1}{q - 1}} |||u_0|||_h^{\frac{ph}{k_h}}, \tag{1.4.12}$$

où $k_h = N(p - 2) + ph$.

Notre second résultat principal est le suivant

Théorème 1.4.3 (Non-existence de solutions globales)
Pour $p > 1$ *et* $0 < \gamma < 1$, *considérons*

$$q_c := p - 1 + (1 - \gamma)(p - 2) + \frac{(2 - \gamma)p(1 + (1 - \gamma)(p - 2))}{n - (1 - \gamma)p}.$$

Si $p - 1 + (1 - \gamma)(p - 2) < q \leq q_c$, *alors le problème* (1.4.1) *n'a pas de solutions faibles globales non-triviales.*

La méthode des fonctions tests constitue aussi l'idée principale de la preuve de ce résultat.

1.5 Durée de vie des solutions non-négatives pour des équations d'évolution non-locales en temps

Dans cette partie, nous étudions l'intervalle maximal d'existence de solution du problème

$$u_t - \Delta |u|^{m-1}u = \int_0^t (t-s)^{-\gamma} |u|^p u(s)\, ds, \quad x \in \mathbb{R}^N,\ t > 0, \tag{1.5.1}$$

où $p > 0$, $m > 1$ et $0 < \gamma < 1$ avec une condition initiale non triviale non négative continue bornée

$$u(x,0) = u_0(x) \not\equiv 0, \quad u_0(x) \geq 0,\ x \in \mathbb{R}^N. \tag{1.5.2}$$

L'équation (1.5.1) sans le terme non linéaire, c.à.d $u_t - \Delta |u|^{m-1}u = 0$, est dite équation des milieux poreux, elle a fait l'objet de plusieurs études. Actuellement, de nombreux résultats intéressants ont été obtenus [63, 67].
Il y a beaucoup d'applications physiques décrites par ce modèle d'une manière naturelle, principalement les processus impliquant l'écoulement d'un fluide, le transfert de chaleur ou la diffusion. Peut-être la plus connue d'entre elles est la description de l'écoulement d'un gaz isotrope à travers un milieu poreux, modélisé par Muskat [56] autour de 1930. D'autres applications ont été proposées dans la biologie mathématique, la propagation de fluides visqueux, et d'autres domaines.
Quand l'équation (1.5.1) est considérée avec le terme non linéaire de la forme $|u|^p u$

$$u_t - \Delta |u|^{m-1}u = |u|^p u,$$

qui est un cas particulier de (1.5.1), elle correspond au cas où $\gamma \to 1$. Cette équation a été considérée par H. J. Kuiper [41]. Nous désignons par durée de vie $L(\sigma)$ pour la condition initiale u_0, la borne supérieure de toutes les valeurs T tel que $[0, T)$ est un intervalle maximal d'existence d'une solution. Kuiper a trouvé que la durée de vie $L(\sigma)$ est bornée par $C\sigma^{-(p+1-m)}$ quand $u_\sigma(x,0) = \sigma u_0(x)$, $\sigma > 0$. (voir Théorème 3.6 [41]).
Cette partie est motivée mathématiquement par les articles récents [10, 47] qui traitent l'exposant critique et la durée de vie pour l'équation parabolique avec un terme non linéaire non local en temps

$$u_t - \Delta u = \int_0^t (t-s)^{\alpha-1} |u|^p u\, ds, \quad x \in \mathbb{R}^N,\ t > 0, \tag{1.5.3}$$

où $u_0 \in C_0(\mathbb{R}^N)$, l'espace de toutes les fonctions continues qui tendent vers zero à l'infini. Cette équation est un cas particulier de (1.5.1) correspondant à $m \to 1$.
Notre analyse est basée sur le fait que l'équation (1.5.1) peut être écrite sous la forme suivante :

$$u_t - \Delta |u|^{m-1}u = J^{\alpha}_{0|t}(|u|^p). \tag{1.5.4}$$

L'intégrale $J^{\alpha}_{0|t}$ joue un rôle essentiel dans la démonstration du théorème sur la durée de vie. Notons que $\alpha = 1 - \gamma \in (0,1)$ dans (1.5.4).
Notre premier résultat est le suivant

Théorème 1.5.1 (Non-existence de solution globale)
Soient $p > 0$, $m > 1$, $0 < \gamma < 1$ et $\alpha = 1 - \gamma$. Si

$$p \leq p^* = m - 1 + \frac{N\alpha(m-1) + 2\alpha m + 2}{(N - 2\alpha)_+},$$

alors le problème (1.5.1)-(1.5.2) n'a pas de solution globale non-triviale.

Supposons que u_σ est la solution correspondant à la donnée initiale non-négative non-triviale $u_\sigma(x,0) = \sigma u_0(x)$. Soit $[0, T_\sigma)$ l'intervalle maximal d'existence. Notre second résultat est le

Théorème 1.5.2 (Borne sur la durée de vie $L(\sigma)$)
Soit u_0 une fonction non négative non-triviale continue dans $\mathbb{R}^N$. Alors, il existe des constantes positives Λ_m, C et σ_1 de sorte que la durée de vie $L(\sigma)$ correspondant à la donnée initiale σu_0 avec $\sigma > \Lambda_m$ satisfait à l'inégalité

$$L(\sigma) \leq C\sigma^{-\frac{p+1-m}{\alpha+1}}. \tag{1.5.5}$$

Nous montrons une condition nécessaire pour la solution globale pour l'équation (1.5.1). Plus précisément, si u est une solution globale de l'équation (1.5.1) avec donnée initiale $u(x,0) = u_0(x)$, alors une inégalité de la forme suivante

$$\limsup_{R\to\infty} R^{-S} \int_{B_R} u_0(x)\varphi_1(x/R)\, dx \leq C\lambda^\kappa, \qquad \text{pour } S > 0 \text{ et } \kappa > 0,$$

est satisfaite. La fonction φ_1 introduite ci-dessus est la fonction propre positive correspondante à la valeur propre principale du problème de Dirichlet sur la boule unité B_1, et qui est normalisée d'une maniére que $\int_{B_1} \varphi_1(\xi)\, d\xi = 1$. Les constantes C et κ dépendent de N, m, p et α.

La méthode utilisée pour prouver la durée de vie et les conditions nécessaires pour l'existence de solution locale et globale est la méthode de fonctions tests [3, 4, 23, 54, 70]. Le principe de cette méthode est le suivant : nous supposons, par l'absurde, que la solution est globale. Ensuite, nous faisons un choix approprié de la fonction test.

Chapitre 2

Blow-up of solutions for a Semilinear Fractional Schrödinger equation

Ihab DANNAWI, Ahmad Z. FINO and Mokhtar KIRANE

Abstract

In this article, we consider the Cauchy problem in $\mathbb{R}^N$, $N \geq 1$, for a semi-linear Schrödinger equation. We recall the local well-posedness of solutions in $H^{\frac{\alpha}{2}}(\mathbb{R}^N)$, $0 < \alpha \leq 2$. We mainly study the global behavior of the solutions, and we will prove a blow-up result under some conditions on the data.

Keywords : Schrödinger equations, Blow-up, Fractional Laplacian.
MSC : 35Q55; 35B44;

2.1 Introduction

We study the initial-value problem for the nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u = \Lambda^\alpha u + \lambda |u|^p & (t,x) \in [0,T) \times \mathbb{R}^N, \\ \\ u(x,0) = f(x) & x \in \mathbb{R}^N, \end{cases} \tag{2.1.1}$$

where the pseudo-differential operator $\Lambda^\alpha = (-\Delta)^{\alpha/2}$ with $0 < \alpha \leq 2$ is defined via the Fourier transformation : $\widehat{\Lambda^\alpha u}(\xi) = |\xi|^\alpha \widehat{u}(\xi)$. Moreover, we assume that $T > 0$, $1 < p \leq 1 + \frac{\alpha}{N}$, $u = u(x,t)$ is a complex-valued unknown function, $\lambda \in \mathbb{C}\backslash\{0\}$ and $f = f(x) \in H^{\frac{\alpha}{2}}(\mathbb{R}^N)$ is a given complex-valued function.

In recent years, the study of fractional calculus and fractional integrodifferential equations applied to physics and other areas has grown considerably, (see [37], [52],[53]) and references therein.

Meltzler and Klafter discussed recent developments in the description of anomalous diffusion by the fractional dynamics approach in [52] and [53] where many fractional partial differential equations are derived asymptotically from Lévy random walk models, a natural generalization of the brownian walk models. Inspired by the Feynman path approach to quantum mechanics, Laskin used the path integral over Lévy -like quantum mechanical paths to obtain a fractional Schrödinger equation, which extends a classical result that the path integral over brownian trajectories leads to the standard Schrödinger equations, (see [43],[44]). There are some papers in studying fractional Schrödinger equations and their applications, (see [26],[62])and references therein.
When $\alpha = 2$, it is well known that local well-posedness holds for (2.1.1) in $H^1(\mathbb{R}^N)$ if $1 < p < 1 + \frac{4}{(N-2)_+}$ (see [11]). Moreover, it is also known that the local solutions can be extended globally for some small data when p is larger than the Strauss exponent p_s, which is the positive root of $Np^2 - (N+2)p - 2 = 0$ (see [9]). However, there have been no results on global existence for $p \leq p_s$. In 2013 Ikeda and Wakasugi [13] have proved a small-data blow-up result for (1) when $1 < p \leq 1 + \frac{2}{N}$.

The main goal in this paper is to generalize the result of blow-up of Ikeda and Wakasugi [13] to the fractional Schrödinger equations (2.1.1).

The local existence is done by the Banach fixed point theorem, using the semigroup theory and stone's theorem on the fractional operator $A = -i(-\Delta)^{\frac{\alpha}{2}}$ which is the infinitesimal generator of a C_0 group of unitary operator $S(t) = e^{-i(-\Delta)^{\frac{\alpha}{2}}}$ on L^2 (see [11]).

The method used to prove the blow-up result is the method of the test function initiated by Baras and Pierre [4], then developed by Zhang [70], and later by Mitidieri and Pohozaev [55].

The paper is organised as follows. In Section 2, we present the local existence result of solutions for (2.1.1) with some properties. Section 3 contains the blow-up result of solutions for (2.1.1).

2.2 Local existence

This section is dedicated to proving the local existence and uniqueness of mild solutions to the problem (2.1.1). Let $Au = -i(-\Delta)^{\frac{\alpha}{2}}u$. Applying Stone's Theorem [61, Theorem 1.10.8], we know that A is the infinitesimal generator of a C_0 group of unitary operator $S(t)$, $-\infty < t < \infty$, on $L^2(\mathbb{R}^N)$. We start by giving the

Definition 2.2.1 (Mild solution) *Let $f \in H^{\frac{\alpha}{2}}(\mathbb{R}^N)$, $0 < \alpha \leq 2$, $p > 1$ and $T > 0$. We say that $u \in C([0,T], H^{\frac{\alpha}{2}}(\mathbb{R}^N))$ is a mild solution of the problem (2.1.1) if u satisfies the following integral equation*

$$u(t) = S(t)f - i\lambda \int_0^t S(t-s)|u(s)|^p \, ds. \tag{2.2.1}$$

Theorem 2.2.2 (Local existence)
Given $f \in H^{\frac{\alpha}{2}}(\mathbb{R}^N)$, $\lambda \in \mathbb{C}\backslash\{0\}$, and $1 < p \leq 1+\frac{2\alpha}{(N-\alpha)_+}$, then there exist a maximal time $T_{\max} > 0$ and a unique mild solution $u \in C([0,T_{\max}), H^{\frac{\alpha}{2}}(\mathbb{R}^N))$ of the problem (2.1.1). Furthermore, either $T_{\max} = \infty$ or else $T_{\max} < \infty$ and $\|u\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^N)} \to \infty$ as $t \to T_{\max}$.

Proof. By the Banach fixed point theorem, we obtain the existence of a unique mild solution $u \in \Pi_T := C([0,T], H^{\frac{\alpha}{2}}(\mathbb{R}^N))$ to the problem (2.1.1) [12, Section 4].
Using the uniqueness of solutions, we conclude the existence of a solution on a maximal interval $[0, T_{\max})$ where

$$T_{\max} := \sup\{T > 0\ ;\ \text{there exists a mild solution } u \in \Pi_T \text{ to (2.1.1)}\} \leq +\infty.$$

Next we prove that $\|u\|_{H^{\frac{\alpha}{2}}} \to \infty$ as $t \to T_{\max}$. We suppose

$$\liminf_{t\to T_{max}} \|u\|_{H^{\frac{\alpha}{2}}} < \infty.$$

Then we can find a sequence $\{t_k\}_{k\in\mathbb{N}} \subset [0, T_{\max})$ and a positive constant $M > 0$ such that

$$\lim_{k\to\infty} t_k = T_{max} \tag{2.2.2}$$

and

$$\sup_{k\in\mathbb{N}} \|u(t_k)\|_{H^{\frac{\alpha}{2}}} \leq M. \tag{2.2.3}$$

By (2.2.3) and Theorem 2.2.2, we can construct a solution $u \in C([t_k, t_k + T(M)); H^{\frac{\alpha}{2}}(\mathbb{R}^N))$ of (2.2.1) for all $k \in \mathbb{N}$ with some $T(M) > 0$. However, by (2.2.2), we can take t_k satisfying $t_k + T(M) > T_{max}$, which contradicts the definition of T_{max}. Therefore, we obtain

$$\liminf_{t\to T_{max}} \|u\|_{H^{\frac{\alpha}{2}}} = \infty.$$

□

2.3 Blow-up of solutions

Now, we want to derive a blow-up result for Eq. (2.1.1). Our argument uses weak solutions.

Definition 2.3.1 (Weak solution) *Let $f \in H^{\frac{\alpha}{2}}(\mathbb{R}^N)$, $0 < \alpha \leq 2$ and $T > 0$. We say that u is a weak solution of the problem (2.1.1) if $u \in C([0,T]; L^p(\mathbb{R}^N))$ and verifies the equation*

$$\begin{aligned} i\int_{\mathbb{R}^N} f(x)\varphi(x,0) + \lambda\int_0^T\int_{\mathbb{R}^N} |u|^p\varphi(x,t) = &-\int_0^T\int_{\mathbb{R}^N} u(x,t)(-\Delta)^{\alpha/2}\varphi(x,t) \\ &- i\int_0^T\int_{\mathbb{R}^N} u(x,t)\varphi_t(x,t), \end{aligned} \tag{2.3.1}$$

for all $\varphi \in C_0^2([0,T]\times\mathbb{R}^N)$ such that $\varphi(\cdot\,,T) = 0$.

Lemma 2.3.2 *Consider $f \in H^{\frac{\alpha}{2}}(\mathbb{R}^N)$ and let $u \in C([0,T], H^{\frac{\alpha}{2}}(\mathbb{R}^N))$ be a mild solution of (2.1.1), then u is a weak solution of (2.1.1), for all $0 < \alpha \leq 2$ and all $T > 0$.*

Proof. Let $T > 0$, $0 < \alpha \leq 2$, $f \in H^{\frac{\alpha}{2}}(\mathbb{R}^N)$ and let $u \in C([0,T], H^{\frac{\alpha}{2}}(\mathbb{R}^N))$ be a solution of (2.2.1). Given $\varphi \in C^1([0,T], H^{\frac{\alpha}{2}}(\mathbb{R}^N))$ with compact support such that $\varphi(\cdot\,, T) = 0$. Then after multiplying (2.2.1) by φ and integrating over $\mathbb{R}^N$, we obtain

$$\int_{\mathbb{R}^N} u(x,t)\varphi(x,t) = \int_{\mathbb{R}^N} S(t)f(x)\varphi(x,t) - i\lambda \int_{\mathbb{R}^N} \int_0^t S(t-s)|u(s)|^p\, ds\varphi(x,t).$$

We differentiate to obtain

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^N} u(x,t)\varphi(x,t) &= \int_{\mathbb{R}^N} \frac{d}{dt}\left(S(t)f(x)\varphi(x,t)\right) \\ &- i\lambda \int_{\mathbb{R}^N} \frac{d}{dt}\int_0^t S(t-s)|u(s)|^p\, ds\varphi(x,t). \end{aligned} \tag{2.3.2}$$

Now, using that $(-\Delta)^{\alpha/2}$ is a self-adjoint operator with $D((-\Delta)^{\alpha/2}) = H^\alpha(\mathbb{R}^N)$ and a property of the group $S(t)$ ([9, Chapter 3]), we have :

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{d}{dt}\left(S(t)f(x)\varphi(x,t)\right) &= \int_{\mathbb{R}^N} A\left(S(t)f(x)\right)\varphi(x,t) + \int_{\mathbb{R}^N} S(t)f(x)\varphi_t(x,t) \\ &= \int_{\mathbb{R}^N} S(t)f(x)A\varphi(x,t) + \int_{\mathbb{R}^N} S(t)f(x)\varphi_t(x,t), \end{aligned} \tag{2.3.3}$$

and

$$\begin{aligned} i\lambda \int_{\mathbb{R}^N} \frac{d}{dt}\int_0^t S(t-s)|u(s)|^p\, ds\varphi(x,t) &= i\lambda \int_{\mathbb{R}^N} |u(s)|^p\varphi(x,t) \\ &+ i\lambda \int_{\mathbb{R}^N}\int_0^t A\left(S(t-s)|u(s)|^p\right)\, ds\varphi(x,t) \\ &+ i\lambda \int_{\mathbb{R}^N}\int_0^t S(t-s)|u(s)|^p\, ds\varphi_t(x,t) \\ &= i\lambda \int_{\mathbb{R}^N} |u(s)|^p\varphi(x,t) \\ &+ i\lambda \int_{\mathbb{R}^N}\int_0^t S(t-s)|u(s)|^p\, dsA\varphi(x,t) \\ &+ i\lambda \int_{\mathbb{R}^N}\int_0^t S(t-s)|u(s)|^p\, ds\varphi_t(x,t). \end{aligned} \tag{2.3.4}$$

Thus, using (2.2.1), (2.3.3) and (2.3.4), we conclude that (2.3.2) implies

$$\frac{d}{dt}\int_{\mathbb{R}^N} u(x,t)\varphi(x,t) = \int_{\mathbb{R}^N} u(x,t)A\varphi(x,t) + \int_{\mathbb{R}^N} u(x,t)\varphi_t(x,t) - i\lambda \int_{\mathbb{R}^N} |u(s)|^p\varphi(x,t).$$

We conclude by integrating in time over $[0,T]$ and using the fact that $\varphi(\cdot\,,T) = 0$. □

To state our result, we put $\lambda = \lambda_1 + i\lambda_2$ and $f = f_1 + if_2$. We introduce the following assumption on the data :

$$\begin{aligned} & f_1 \in L^1(\mathbb{R}^N), \quad \lambda_2 \int_{\mathbb{R}^N} f_1 \, dx > 0, \\ & \textit{or} \\ & f_2 \in L^1(\mathbb{R}^N), \quad \lambda_1 \int_{\mathbb{R}^N} f_2 \, dx < 0. \end{aligned} \tag{2.3.5}$$

Theorem 2.3.3 *Let $\lambda \in \mathbb{C}\backslash\{0\}$, and $f \in H^{\frac{\alpha}{2}}(\mathbb{R}^N)$ satisfying (2.3.5). If $1 < p \leq 1 + \frac{\alpha}{N}$ then the mild solution of (2.1.1) blows-up in a finite time.*

Proof. Let on the contrary u be a global weak solution to (2.1.1), then $u \in L^p((0, R^\alpha), L^p(B_{2\rho}))$ for $R \in \mathbb{R}_+^*$ and $\rho > 0$, where $B_{2\rho}$ stands for the closed ball of center 0 and radius 2ρ. Let us define the function $\varphi(x,t) := \varphi_1(x/BR)\,(\varphi_2(t))^\ell$, where $\ell = \frac{2p-1}{p-1}$, $R, B > 0$ and $0 \leq \varphi_1 \in D(\Delta_D^{\alpha/2})$ is the first eigenfunction of the fractional Laplacian operator $\Delta_D^{\alpha/2}$ in B_2, with the homogeneous Dirichlet boundary condition, associated to the first eigenvalue $\kappa := \kappa_1^{\alpha/2}$, and $\varphi_2(t) = \psi\left(\frac{t}{R^\alpha}\right)$, where ψ is a smooth non-increasing function on $[0,\infty)$ such that

$$\psi(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq 1, \\ 0 & \text{if } r \geq 2. \end{cases}$$

The constant $B > 0$ in the definition of φ_1 is fixed and will be chosen later. In fact, it plays a role in the critical case $p = 1 + \alpha/N$. Only while in the subcritical case $p < 1 + \alpha/N$, we simply put $B = 1$.

In the following, Ω_1 and Ω_2 denote the supports of φ_1 and φ_2, respectively :

$$\Omega_1 = \left\{x \in \mathbb{R}^N \ : \ |x| \leq 2BR\right\}, \qquad \Omega_2 = \{t \in [0,\infty) \ : \ t \leq 2R^\alpha\}.$$

As u is a weak solution, we have

$$\begin{aligned} \lambda \int_{\Omega_2}\int_{\Omega_1} |u|^p(x,t)\varphi(x,t)\,dxdt + i\int_{\Omega_1} f(x)\varphi(x,0)\,dx & \\ = \quad -i\int_{\Omega_2}\int_{\Omega_1} u(x,t)\varphi_1(x/BR)\partial_t\varphi_2^\ell(t)\,dxdt & \\ + \int_{\Omega_2}\int_{\Omega_1} u(x,t)\varphi_2^\ell(t)\Lambda^\alpha(\varphi_1(x/BR))\,dxdt. & \end{aligned} \tag{2.3.6}$$

We consider four cases :

Case I : If $\lambda_1 > 0$, then $\int_{\mathbb{R}^N} f_2\,dx < 0$, therefore taking (Re) of both sides of (2.3.6), we get :

$$\begin{aligned} \lambda_1 \int_{\Omega_2}\int_{\Omega_1} |u|^p(x,t)\varphi(x,t)\,dxdt - \int_{\Omega_1} f_2(x)\varphi(x,0)\,dx & \\ = \quad \mathrm{Re}\big[-i\int_{\Omega_2}\int_{\Omega_1} u(x,t)\varphi_1(x/BR)\partial_t\varphi_2^\ell(t)\,dxdt\big] & \end{aligned}$$

$$+ \quad \mathrm{Re}\big[\int_{\Omega_2}\int_{\Omega_1} u(x,t)\varphi_2^\ell(t)\Lambda^\alpha(\varphi_1(x/BR))\,dxdt\big].$$

The left hand side of the above equation is nonnegative.
Case II : If $\lambda_1 < 0$, then $\int_{\mathbb{R}^N} f_2\,dx > 0$, therefore taking (-Re) part of both sides of (2.3.6), we get :

$$\begin{aligned} -\lambda_1 \int_{\Omega_2}\int_{\Omega_1} |u|^p(x,t)\varphi(x,t)\,dxdt &+ \int_{\Omega_1} f_2(x)\varphi(x,0)\,dx \\ &= \mathrm{Re}\big[i\int_{\Omega_2}\int_{\Omega_1} u(x,t)\varphi_1(x/BR)\partial_t\varphi_2^\ell(t)\,dxdt\big] \\ &+ (\text{-Re})\big[\int_{\Omega_2}\int_{\Omega_1} u(x,t)\varphi_2^\ell(t)\Lambda^\alpha(\varphi_1(x/BR))\,dxdt\big]. \end{aligned}$$

The left hand side of the above equation is nonnegative.
Case III : If $\lambda_2 > 0$, then $\int_{\mathbb{R}^N} f_1\,dx > 0$, therefore taking (Im) part of both sides of (2.3.6), we get :

$$\begin{aligned} \lambda_2 \int_{\Omega_2}\int_{\Omega_1} |u|^p(x,t)\varphi(x,t)\,dxdt &+ \int_{\Omega_1} f_1(x)\varphi(x,0)\,dx \\ &= \mathrm{Im}\big[-i\int_{\Omega_2}\int_{\Omega_1} u(x,t)\varphi_1(x/BR)\partial_t\varphi_2^\ell(t)\,dxdt\big] \\ &+ \mathrm{Im}\big[\int_{\Omega_2}\int_{\Omega_1} u(x,t)\varphi_2^\ell(t)\Lambda^\alpha(\varphi_1(x/BR))\,dxdt\big]. \end{aligned}$$

The left hand side of the above equation is nonnegative.
Case IV : If $\lambda_2 < 0$, then $\int_{\mathbb{R}^N} f_1\,dx < 0$, therefore taking (-Im) part of both sides of (2.3.6), we get :

$$\begin{aligned} -\lambda_2 \int_{\Omega_2}\int_{\Omega_1} |u|^p(x,t)\varphi(x,t)\,dxdt &- \int_{\Omega_1} f_1(x)\varphi(x,0)\,dx \\ &= \mathrm{Im}\big[i\int_{\Omega_2}\int_{\Omega_1} u(x,t)\varphi_1(x/BR)\partial_t\varphi_2^\ell(t)\,dxdt\big] \\ &+ \mathrm{Im}\big[\int_{\Omega_2}\int_{\Omega_1} u(x,t)\varphi_2^\ell(t)\Lambda^\alpha(\varphi_1(x/BR))\,dxdt\big]. \end{aligned}$$

The left hand side of the above equation is nonnegative.
Then we only consider the Case I, since the others can be treated in the same way. In this case we may assume $f_2 \in L^1$ and

$$\int_{\mathbb{R}^N} f_2\,dx < 0.$$

Thus we have :

$$\lambda_1 \int_{\Omega_2}\int_{\Omega_1} |u|^p(x,t)\varphi(x,t)\,dxdt$$

$$\begin{aligned} &\leq \ \ell\kappa B^{-\alpha}\int_{\Omega_2}\int_{\Omega_1} u(x,t)\varphi_2^{\ell}(t)R^{-\alpha}\varphi_1(x/BR)\,dxdt. \\ &+ \ \mathrm{Im} \ \ \ell\int_{\Omega_2}\int_{\Omega_1} u(x,t)\varphi_1(x/BR)\varphi_2^{\ell-1}(t)\partial_t\varphi_2(t)\,dxdt := I_2 + I_1. \end{aligned} \tag{2.3.7}$$

In (2.3.7), we have used the fact that $\Delta_D^{\alpha/2}\varphi_1(x/BR) = R^{-\alpha}B^{-\alpha}\kappa\varphi_1(x/R)$.
Hence, by the ε-Young inequality $ab \leq \varepsilon a^p + C(\varepsilon)b^{\ell-1}$ (note that $1/p + 1/(\ell-1) = 1$) with $\varepsilon > 0, a > 0$ and $b > 0$, we deduce :

$$\begin{aligned} I_1 &= \mathrm{Im} \ \ \ell\int_{\Omega_2}\int_{\Omega_1} u(x,t)\varphi^{\frac{1}{p}}\varphi^{-\frac{1}{p}}\varphi_1(x/BR)\varphi_2^{\ell-1}(t)\partial_t\varphi_2(t)\,dxdt \\ &\leq \frac{\lambda_1}{4}\int_{\Omega_2}\int_{\Omega_1} |u|^p(x,t)\varphi(x,t)\,dxdt \\ &+ \ C\int_{\Omega_2}\int_{\Omega_1} \varphi^{-\frac{\ell-1}{p}}\varphi_1^{(\ell-1)}(x/BR)\varphi_2^{(\ell-1)^2}(t)|\partial_t\varphi_2(t)|^{\ell-1}\,dxdt \\ &\leq \frac{\lambda_1}{4}\int_{\Omega_2}\int_{\Omega_1} |u|^p(x,t)\varphi(x,t)\,dxdt \\ &+ \ C\int_{\Omega_2}\int_{\Omega_1} \varphi_1(x/BR)\varphi_2(t)|\partial_t\varphi_2(t)|^{\ell-1}\,dxdt, \end{aligned}$$

and

$$\begin{aligned} I_2 &= \ell\lambda B^{-\alpha}\int_{\Omega_2}\int_{\Omega_1} u(x,t)\varphi^{\frac{1}{p}}\varphi^{-\frac{1}{p}}\varphi_2^{\ell}(t)R^{-\alpha}\varphi_1(x/BR)\,dxdt \\ &\leq \frac{\lambda_1}{4}\int_{\Omega_2}\int_{\Omega_1} |u|^p(x,t)\varphi(x,t)\,dxdt \\ &+ \ C\int_{\Omega_2}\int_{\Omega_1} \varphi^{-\frac{\ell-1}{p}}\varphi_2^{\ell(\ell-1)}(t)R^{-\alpha(\ell-1)}\varphi_1^{(\ell-1)}(x/BR)\,dxdt \\ &\leq \frac{\lambda_1}{4}\int_{\Omega_2}\int_{\Omega_1} |u|^p(x,t)\varphi(x,t)\,dxdt \\ &+ \ C\int_{\Omega_2}\int_{\Omega_1} \varphi_2^{\ell-1}(t)R^{-\alpha(\ell-1)}\varphi_1(x/BR)\,dxdt. \end{aligned}$$

whereupon

$$\begin{aligned} &\frac{\lambda_1}{2}\int_{\Omega_2}\int_{\Omega_1} |u|^p(x,t)\varphi(x,t)\,dxdt \\ &\qquad\leq \ C\int_{\Omega_2}\int_{\Omega_1} \varphi_1(x/BR)\varphi_2(t)|\partial_t\varphi_2(t)|^{\ell-1}\,dxdt \\ &\qquad+ \ C\int_{\Omega_2}\int_{\Omega_1} \varphi_1(x/BR)\varphi_2^{\ell-1}(t)R^{-\alpha(\ell-1)}\,dxdt. \end{aligned} \tag{2.3.8}$$

Recall now that φ_1 and φ_2 depend on $R > 0$. Hence changing the variables $\xi = (BR)^{-1}x$ and $\tau = R^{-\alpha}t$, we obtain from (2.3.8)

$$\frac{\lambda_1}{2}\int_{\Omega_2}\int_{\Omega_1} |u|^p(x,t)\varphi(x,t)\,dxdt$$

$$
\begin{aligned}
&\leq \; C\int_0^2\int_{|\xi|\leq 2}\varphi_1(\xi)\varphi_2(\tau)R^{-\alpha(\ell-1)}|\partial_t\varphi_2(\tau)|^{\ell-1}R^NR^\alpha\,d\xi d\tau\\
&+ \; C\int_0^2\int_{|\xi|\leq 2}\varphi_1(\xi)\varphi_2^{\ell-1}(\tau)R^{-\alpha(\ell-1)}B^NR^NR^\alpha\,d\xi d\tau.
\end{aligned}
$$

Therefore, we easily get :

$$
\frac{\lambda_1}{2}\int_{\Omega_2}\int_{\Omega_1}|u|^p(x,t)\varphi(x,t)\,dxdt \leq CR^{N+\alpha-\alpha(\ell-1)}, \tag{2.3.9}
$$

where the constant C in the right hand side of (2.3.9) is independent of R. Note that $N+\alpha-\alpha(\ell-1)\leq 0$ if and only if $p\leq 1+\alpha/N$. Now, we consider two cases.

For $p<1+\alpha/N$, we have $N+\alpha-\alpha(\ell-1)<0$. Hence, taking the limit $R\to\infty$ in (2.3.9) and using the Lebesgue dominated convergence theorem, we obtain

$$
\frac{\lambda_1}{2}\int_0^\infty\int_{\mathbb{R}^N}|u|^p(x,t)\varphi_1(0)\,dxdt = \lim_{R\to\infty}\int_0^R\int_{\mathbb{R}^N}|u|^p(x,t)\varphi(x,t)\,dxdt\leq 0
$$

Then $u(x,t)=0$ for all x and t.

In the critical case $p=1+\alpha/N$, we estimate the first term in the right hand side of inequality (2.3.7) using again the ε-Young inequality, and estimate the second term by the Hölder inequality (with $\bar{p}=p/(p-1)=\ell-1$) as follows

$$
\begin{aligned}
&\lambda_1\int_{\Omega_2}\int_{\Omega_1}|u|^p\varphi(x,t)\,dxdt\\
&\leq \frac{\lambda_1}{2}\int_{\Omega_2}\int_{\Omega_1}|u|^p\varphi(x,t)\,dxdt\\
&\quad +C\int_{\Omega_2}\int_{\Omega_1}\varphi^{-\frac{\bar{p}}{p}}\varphi_2^{\ell\bar{p}}(t)\varphi_1^{\bar{p}}(x/BR)(RB)^{-\alpha\bar{p}}\,dxdt\\
&\quad +\ell\left(\int_{\Omega_3}\int_{\Omega_1}|u|^p(x,t)\,dxdt\right)^{1/p}\\
&\quad\times\left(\int_{\Omega_2}\int_{\Omega_1}\varphi_1^{\bar{p}}(x/BR)\varphi_2^{(\ell-1)\bar{p}}(t)\,|\partial_t\varphi_2(t)|^{\bar{p}}\,dxdt\right)^{1/\bar{p}}.
\end{aligned}\tag{2.3.10}
$$

Here, $\Omega_3=\{t\in[0,\infty)\;:\;R^\alpha\leq t\leq 2R^\alpha\}$ is the support of $\partial_t\varphi_2$. Note that

$$
\begin{aligned}
&\lim_{R\to\infty}\int_{\Omega_1}\int_{\Omega_3}|u|^p(x,t)\,dx\,dt\\
=\;&\lim_{R\to\infty}\int_{\Omega_1}\int_{|t|\leq 2R^\alpha}|u|^p(x,t)\,dx\,dt-\lim_{R\to\infty}\int_{\Omega_1}\int_{|t|\leq R^\alpha}|u|^p(x,t)\,dx\,dt\\
=\;&\int_0^\infty\int_{\mathbb{R}^N}|u|^p(x,t)\,dx\,dt-\int_0^\infty\int_{\mathbb{R}^N}|u|^p(x,t)\,dx\,dt=0
\end{aligned}\tag{2.3.11}
$$

because $u \in L^p(\mathbb{R}^N \times [0,\infty))$ (cf. (2.3.9)).
Now, introducing the new variables $\xi = (BR)^{-1}x$, $\tau = R^{-\alpha}t$ and recalling that $p = 1+\alpha/N$, we rewrite (2.3.11) as follows

$$\begin{aligned}
&\frac{\lambda_1}{2}\int_{\Omega_2}\int_{\Omega_1}|u|^p\varphi(x,t)\,dxdt \\
&\leq C\int_0^2\int_{|\xi|\leq 2B}\varphi_2^\ell(\tau)\varphi_1(\xi)B^{-\alpha}\,d\xi d\tau \\
&+\ell\left(\int_{\Omega_3}\int_{\Omega_1}|u|^p(x,t)\,dxdt\right)^{1/p}\times\left(\int_0^2\int_{|\xi|\leq 2B}\varphi_2^\ell(\tau)\varphi_1(\xi)B^N\,|\partial_t\varphi_2(\tau)|^{\bar{p}}\,d\xi d\tau\right)^{\frac{1}{\bar{p}}} \\
&\leq C_2B^{-\alpha}+C_1B^{N/\bar{p}}\left(\int_{\Omega_3}\int_{\Omega_1}|u|^p(x,t)\,dxdt\right)^{1/p},
\end{aligned} \tag{2.3.12}$$

where the constants C_1, C_2 are independent of R and B. Passing in (2.3.12) to the limit as $R \to +\infty$, using (2.3.11) and the Lebesgue dominated convergence theorem we get

$$\frac{\lambda_1}{2}\int_0^\infty\int_{\mathbb{R}^N}|u|^p\varphi(x,t)\,dxdt \leq C_2B^{-\alpha}. \tag{2.3.13}$$

Note that we choose $1 \leq B < R$ large enough such that when $R \to \infty$ we don't have $B \to \infty$ at the same time.
Finally, computing the limit as $B \to \infty$ in (2.3.13), we infer that $u(x,t) = 0$.

□

Chapitre 3

Finite time blow-up for a damped wave equation with space-time dependent potential and nonlinear memory

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Abstract

In this paper, we consider the Cauchy problem in $\mathbb{R}^n$, $n \geq 1$, for a semilinear damped wave equation with space-time dependent potential and nonlinear memory. A blow-up result under a condition on the data in any dimensional space is obtained. Moreover, the local existence in the energy space is also studied.

Keywords : Nonlinear damped wave equation, Blow-up, Subcritical potential
MSC : 35L15; 35L70; 35B33; 34B44;

3.1 Introduction

This paper concerns the Cauchy problem for the following semilinear damped wave equation

$$\begin{cases} u_{tt} - \Delta u + a(x)b(t)u_t = \displaystyle\int_0^t (t-s)^{-\gamma}|u(s)|^p\,ds, & t > 0, x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{3.1.1}$$

where the unknown function u is real-valued, $n \geq 1$, $0 < \gamma < 1$, $p > 1$. The coefficient of the damping term is given by

$$a(x)b(t) := a_0(1+|x|^2)^{-\frac{\alpha}{2}}(1+t)^{-\beta},$$

with $a_0 > 0$, $\alpha, \beta \geq 0$, $\alpha + \beta < 1$. Throughout this paper, we assume that the initial data is in the energy space

$$(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n). \tag{3.1.2}$$

Here after, $\|\cdot\|_q$ and $\|\cdot\|_{H^1}$ $(1 \leq q \leq \infty)$ stand for the usual $L^q(\mathbb{R}^n)$-norm and $H^1(\mathbb{R}^n)$-norm, respectively.

The nonlinear nonlocal term can be considered as an approximation (with suitable change of variables) of the nonlinearity of the following semilinear damped wave equation

$$u_{tt} - \Delta u + a(x)b(t)u_t = |u(t)|^p$$

since the limit

$$\lim_{\gamma \to 1} s_+^{-\gamma} = \Gamma(1-\gamma)\delta(s)$$

exists in the distributional sense, where Γ is the Euler gamma function. For the equation

$$u_{tt} - \Delta u + u_t = |u(t)|^p \tag{3.1.3}$$

Li and Zhou [50] proved that if $n \geq 2$, $1 < p \leq 1 + 2/n$ and the data are positive on average, then the local solution of (3.1.3) must blow-up in a finite time. Todorova and Yordanov [65] developed a weighted energy method and determined that the critical exponent of (3.1.3) is

$$p_c = 1 + \frac{2}{n},$$

which is well known as Fujita's critical exponent for the heat equation $u_t - \Delta u = u^p$, $u > 0$ (see [24]). More precisely, they proved small data global existence in the case $p > 1 + 2/n$ and blow-up for all solutions of (3.1.3) with positive in average data in the case $1 < p < 1+2/n$. Later on Zhang [70] showed that the critical exponent $p = 1 + 2/n$ belongs to the blow-up region. We mention that Todorova and Yordanov [65] assumed the data to have compact support and essentially used this property. However, Ikehata and Tanizawa [32] removed this assumption. Ikehata et al. [33] investigated the space-dependent equation

$$u_{tt} - \Delta u + a(x)u_t = |u|^p,$$

where

$$a(x) \equiv a_0(1 + |x|^2)^{-\alpha/2}, \quad |x| \to \infty, \qquad \text{and } 0 \leq \alpha < 1;$$

they proved that the critical exponent of (3.1.3) is given by

$$p_c = 1 + \frac{2}{n - \alpha}$$

by using a refined multiplier method. Their method also depends on the finite propagation speed property. Recently, Nishihara [59] and Lin et al. [48] considered the semilinear wave equation with time-dependent damping

$$u_{tt} - \Delta u + b(t)u_t = |u|^p,$$

where

$$b(t)=b_0(1+t)^{-\beta},\quad \beta\in(-1,1).$$

They proved that the critical exponent is

$$p_c=1+\frac{2}{n}.$$

This shows that, time-dependent coefficients decreasing to zero at infinity of damping term do not influence the critical exponent.
On the other hand, Equation (3.1.1) can be written as

$$u_{tt}-\Delta u+a(x)b(t)u_t=J^{\delta}_{0|t}\left(|u|^p\right)(t). \tag{3.1.4}$$

Let us present our main results.

We first announce the following local well-posedness result.

Proposition 3.1.1 *Let $\alpha\geq 0$, $\beta\in\mathbb{R}$, $1<p\leq n/(n-2)$ for $n\geq 3$, and $p\in(1,\infty)$ for $n=1,2$. Under the assumption (3.1.2) and $\gamma\in(0,1)$, the problem (3.1.1) admits a unique maximal mild solution such that*

$$u\in C([0,T_{\max}),H^1(\mathbb{R}^n))\cap C^1([0,T_{\max}),L^2(\mathbb{R}^n)),$$

where $0<T_{\max}\leq\infty$. Moreover, if $T_{\max}<\infty$, then $\|u(t)\|_{H^1}+\|u_t(t)\|_2\to\infty$ as $t\to T_{\max}$.

Remark 3.1.2 *We say that u is a global solution of (3.1.1) if $T_{\max}=\infty$, while in the case $T_{\max}<\infty$, we say that u blows up in a finite time.*

Now, we set

$$p_c:=1+\frac{3-\gamma}{(n-\alpha-1+\gamma)_+}.$$

As

$$(p_c=n/(n-2))\Longleftrightarrow(\gamma=(n+2(\alpha-2))/n),$$

this imply, in the case when $(n+2(\alpha-2))/n\leq\gamma$, that $p_c\leq n/(n-2)$. We note that

$$p_c\to 1+\frac{2}{n-\alpha}\qquad\text{as }\gamma\to 1,$$

recovering the case studied by Ikehata and Tanizawa [32].
Our main result is

Theorem 3.1.3 (Blow-up result)
Under the assumption $\alpha\beta=0$, we introduce the following exponents

$$p^*:=1+\frac{3-\gamma}{(N-\alpha-1+\gamma)_+}\qquad\textit{and}\qquad \gamma^*:=\frac{N+2(\alpha-2)}{N}.$$

Suppose that $(u_0, u_1) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ *and*

$$\int_{\mathbb{R}^n} a(x)u_0\,dx > 0, \qquad \int_{\mathbb{R}^n} u_1(x)\,dx > 0. \tag{3.1.5}$$

Then the mild solution of the problem (3.1.1) *blows up in a finite time if*

i)

$$\begin{cases} 1 < p \leq \dfrac{N}{N-2} \text{ and } \gamma \geq \gamma^* & N \geq 3, \\ & & \text{and} \quad p \leq p^* \\ p \in (1, \infty) & N = 1, 2. \end{cases}$$

ii) $N \geq 3$, $1 < p \leq N/(N-2)$ *and* $\gamma \leq \gamma^*$.

The test function method (see [21, 22, 23, 38, 54, 55, 70] and the references therein) is the key to prove the blow-up result.

Remark 3.1.4 *We don't know whether there exists a global solution for problem (3.1.1) when* $p > p_c$.

This paper is organized as follows : in Section 3.2, we present some definitions and properties concerning the local existence, and the fractional integrals and derivatives. Section 3.3 contains the proof of the local existence result (Proposition 3.1.1). Finally, we prove the existence of blowing-up solution (Theorem 3.1.3) in Section 3.4.

Throughout this paper, C will denote positive constants that may change from line to line.

3.2 Preliminaries

In this section, we give some preliminary properties that will be used in the proof of Proposition 3.1.1 and Theorem 3.1.3. First, we start by

3.2.1 The linear homogeneous case

We consider the linear homogeneous equation

$$\begin{cases} u_{tt} - \Delta u + a(x)b(t)u_t = 0, & t > 0, x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^n. \end{cases} \tag{3.2.1}$$

It is well known that for any $(u_0, u_1) \in H^2 \times H^1(\mathbb{R}^n)$, there exists a unique strong solution u of (3.2.1) (see [29, Theorem 2.27]). Let us denote by $R(t,s)$ the operator which maps the initial data $(u(s), u_t(s)) \in H^2 \times H^1$ given at the time $s \geq 0$ to the solution $u(t) \in H^2$ at the time $t \geq s$, i.e. the solution u of (3.2.1) is defined by $u(t) = R(t,0)(u_0, u_1)$. The operator $R(t,s)$ can be extended uniquely such that $R(t,s) : H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \longrightarrow C([s,\infty), H^1(\mathbb{R}^n)) \cap C^1([s,\infty), H^1(\mathbb{R}^n))$ (see [69, Appendix]). Moreover, for any $T > 0$, the following estimation

$$\|R(t,s)(u_0,u_1)\|_{H^1} + \|\partial_t(R(t,s)(u_0,u_1))\|_{L^2} \leq C(1+T)\|(u_0,u_1)\|_{H^1 \times L^2} \tag{3.2.2}$$

holds for $s \leq t \leq s + T$.

3.2.2 The linear inhomogeneous case

Let us consider the linear inhomogeneous equation

$$\begin{cases} u_{tt} - \Delta u + a(x)b(t)u_t = F(t,x), & t > 0, x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^n. \end{cases} \tag{3.2.3}$$

Definition 3.2.1 *(Stong solution)*
Let $(u_0, u_1) \in H^2 \times H^1$ and $F \in C([0,\infty); L^2)$. We say that a function u is a strong solution of (3.2.3) if

$$u \in \bigcap_{j=0}^{2} C^{2-j}([0,\infty); H^j),$$

$u(0) = u_0$, $u_t(0) = u_1$, and satisfies the equation (3.2.3) in the sense of $L^2(\mathbb{R}^n)$. It is well known that if $(u_0, u_1) \in H^2 \times H^1$ and $F \in C^1([0,\infty); L^2)$, then there exists a unique strong solution (see [29, Theorem 2.27]).

Definition 3.2.2 *(Mild solution)*
Let $(u_0, u_1) \in H^1 \times L^2$ and $F \in C([0,\infty); L^2)$. We say that a function u is a mild solution of (3.2.3) if $u \in C([0,\infty); H^1) \cap C^1([0,\infty); L^2)$ and u has the initial data $u(0) = u_0$, $u_t(0) = u_1$ and satisfies the integral equation

$$u(t,x) = R(t,0)(u_0,u_1) + \int_0^t S(t,s)F(s,x)\,ds \tag{3.2.4}$$

in the sense of $H^1(\mathbb{R}^n)$, where $S(t,s)g := R(t,s)(0,g)$ for a function $g \in H^1$.

Definition 3.2.3 *(Weak solution)*
Let $(u_0, u_1) \in H^1 \times L^2$ and $F \in C([0,\infty); L^2)$. We say that a function u is a weak solution of (3.2.3) if $u \in C((0,\infty); H^1) \cap C^1([0,\infty); L^2)$ and u has the initial data $u(0) = u_0$, $u_t(0) = u_1$ and satisfies the identity

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^n} u\left(\varphi_{tt} - \Delta\varphi - (a(x)b(t)\varphi)_t\right) dx\,dt \quad &= \quad \int_{\mathbb{R}^n} \left((a(x)u_0(x) + u_1(x))\,\varphi(0,x) - u_0(x)\varphi_t(0,x)\right) dx \\ &\quad + \int_0^\infty \int_{\mathbb{R}^n} F(t,x)\varphi\,dx\,dt \end{aligned} \tag{3.2.5}$$

for any $\varphi \in C_0^\infty([0,\infty) \times \mathbb{R}^n)$.

Proposition 3.2.4 *[69, Proposition 9.14]*
Let $(u_0, u_1) \in H^2 \times H^1$, $F \in C([0,\infty); H^1) \cap C^1([0,\infty); L^2)$ and u be the strong solution of (3.2.3). Then u is a mild solution. Moreover, u satisfies the following energy estimates

$$\|(u_t, \nabla u)(t)\|_{L^2 \times L^2} \le C\|(u_1, \nabla u_0)\|_{L^2 \times L^2} + C\int_0^t \|F(s,\cdot)\|_{L^2}\,ds, \tag{3.2.6}$$

and

$$\|u(t)\|_{L^2} \le C\|(u_0)\|_{L^2} + C\int_0^t \left(\|(u_1, \nabla u_0)\|_{L^2\times L^2} + \int_0^s \|F(\tau,\cdot)\|_{L^2}\, d\tau \right) ds. \tag{3.2.7}$$

Proposition 3.2.5 *[69, Proposition 9.15]*
Let $(u_0, u_1) \in H^1 \times L^2$, $F \in C([0,\infty); L^2)$. Then there exists a unique mild solution u of (3.2.3). Moreover, the mild solution u satisfies the estimates (3.2.6) and (3.2.7).

3.3 Local existence

We start by introducing the definitions of the mild and weak solution of (3.1.1). For nonlinear equations, it is not always true that there exist global-in-time solutions. Therefore, we consider solution defined on an interval $[0, T)$ for $T > 0$. We call such a solution local-in-time solution (or local solution) and if we can take $T = \infty$, then we call it global-in-time solution (or global solution).

Definition 3.3.1 *(Mild solution)*
Let $(u_0, u_1) \in H^1 \times L^2$. We say that a function u is a mild solution of (3.1.1) if

$$u \in C([0, T); H^1) \cap C^1([0, T); L^2),$$

$u(0) = u_0$, $u_t(0) = u_1$, and satisfies the integral equation

$$u(t, x) = R(t, 0)(u_0, u_1) + \Gamma(\delta)\int_0^t S(t, s) J^{\delta}_{0|t}(|u|^p)\, ds \tag{3.3.1}$$

in the sense of $H^1(\mathbb{R}^n)$, where $\delta = 1 - \gamma$.

Definition 3.3.2 *(Weak solution) Let $T > 0$, $\alpha \in [0, 1)$, $\gamma \in (0, 1)$ and $u_0, u_1 \in L^1_{loc}(\mathbb{R}^n)$. We say that u is a weak solution of (3.1.1) if $u \in C([0, T], L^p_{loc}(\mathbb{R}^n))$ and satisfies*

$$\begin{aligned} &\Gamma(\delta)\int_0^T\int_{\mathbb{R}^n} J^{\delta}_{0|t}(|u|^p)\varphi\, dx\, dt + \int_{\mathbb{R}^n} u_1(x)\varphi(0, x)\, dx + \int_{\mathbb{R}^n} u_0(x)(a(x)\varphi(0, x) - \varphi_t(0, x))\, dx \\ &= \int_0^T\int_{\mathbb{R}^n} u\varphi_{tt}\, dx\, dt - \int_0^T\int_{\mathbb{R}^n} u(a(x)b(t)\varphi)_t\, dx\, dt - \int_0^T\int_{\mathbb{R}^n} u\Delta\varphi\, dx\, dt, \end{aligned} \tag{3.3.2}$$

for all compactly supported function $\varphi \in C^2([0, T] \times \mathbb{R}^n)$ such that $\varphi(\cdot, T) = 0$ and $\varphi_t(\cdot, T) = 0$, where $\delta = 1 - \gamma$.

The following lemma is useful for the proof of Theorem 3.1.3.

Lemma 3.3.3 *(Mild $\to$ Weak)*
Let $T > 0$, $\alpha \in [0, 1)$ and $\gamma \in (0, 1)$. Suppose that $1 < p \le n/(n-2)$, if $n \ge 3$, and $p \in (1, \infty)$, if $n = 1, 2$. If $u \in C([0, T], H^1(\mathbb{R}^n)) \cap C^1([0, T], L^2(\mathbb{R}^n))$ is the mild solution of (3.1.1), then u is a weak solution of (3.1.1).

Proof. Let u be a mild solution of (3.1.1) and let $\varphi \in C^2([0,T]\times\mathbb{R}^n)$ be a compactly supported function such that $\varphi(\cdot\,,T)=0$ and $\varphi_t(\cdot\,,T)=0$. Under the assumption that $1<p\le n/(n-2)$, if $n\ge 3$, and $p\in(1,\infty)$, if $n=1,2$, we note that by the Gagliardo-Nirenberg inequality [17, p.3], we have the estimate

$$\begin{aligned}
\|f(t,u)\|_2 &= \left\|\int_0^t (t-s)^{-\gamma}|u(s)|^p\,ds\right\|_2 \le \int_0^t (t-s)^{-\gamma}\|u^p\|_2\,ds = \int_0^t (t-s)^{-\gamma}\|u\|_2^{2p}\,ds\\
&\le C\int_0^t (t-s)^{-\gamma}\|\nabla u\|_2^{\sigma p}\|u\|_2^{(1-\sigma)p}\,ds \le C\int_0^t (t-s)^{-\gamma}\|u\|_{H^1}^p\,ds\\
&\le C\|u\|_{L^\infty((0,T);H^1)}\int_0^t (t-s)^{-\gamma}\,ds\\
&\le C\,T^{1-\gamma}\|u\|_{L^\infty((0,T);H^1)}.
\end{aligned}$$

This inequality shows that $f(u)\in C([0,T];L^2)$. We take sequences $\left\{(u_0^{(j)},u_1^{(j)})\right\}_{j=1}^\infty \subset H^2\times H^1$ and $\left\{(F^{(j)}\right\}_{j=1}^\infty \subset C([0,T];H^1)\cap C^1([0,T];L^2)$ such that

$$\lim_{j\to\infty}(u_0^{(j)},u_1^{(j)})=(u_0,u_1)\ \text{ in }\ H^1\times L^2,\qquad \lim_{j\to\infty}F^{(j)}=f(u)\ \text{ in }\ C([0,T];L^2).$$

Let $u^{(j)}$ be the strong solution of the linear inhomogeneous equation (3.2.3) with the initial data $(u_0^{(j)},u_1^{(j)})$ and the inhomogenous term $F^{(j)}$. Then by Proposition 3.2.4, we have

$$u^{(j)}(t,x)=R(t,0)(u_0^{(j)},u_1^{(j)})+\int_0^t S(t,s)F^{(j)}(s,x)\,ds.$$

Since u is a mild solution of (3.1.1), we obtain

$$u^{(j)}(t,x)-u(t,x)=R(t,0)(u_0^{(j)}-u_0,u_1^{(j)}-u_1)+\int_0^t S(t,s)(F^{(j)}(s,x)-f(u(s,x)))\,ds$$

and hence, by using the estimation (3.2.2), we see that

$$\begin{aligned}
\|(u^{(j)}-u,\partial_t(u^{(j)}-u))(t)\|_{H^1\times L^2} &\le C(1+T)\|(u_0^{(j)}-u_0,u_1^{(j)}-u_1)\|_{H^1\times L^2}\\
&\quad + C(1+T)T\sup_{s\in[0,T]}\|F^{(j)}(s)-f(u(s))\|_{L^2}.
\end{aligned}$$

This implies that $\lim_{j\to\infty}u^{(j)}=u$ in $C([0,T];H^1)\cap C^1([0,T];L^2)$. Since each $u^{(j)}$ is a strong solution of (3.2.3), $u^{(j)}$ is also a weak solution of (3.2.3), that is,

$$\int_0^\infty\int_{\mathbb{R}^n} u^{(j)}\left(\varphi_{tt}-\Delta\varphi-(a(x)b(t)\varphi)_t\right)dx\,dt =$$

$$\int_{\mathbb{R}^n}\left(\left(a(x)u_0^{(j)}(x)+u_1^{(j)}(x)\right)\varphi(0,x)-u_0^{(j)}(x)\varphi_t(0,x)\right)dx\ +\int_0^\infty\int_{\mathbb{R}^n}F^{(j)}(t,x)\varphi\,dx\,dt.$$

Thus, letting $j\to+\infty$, we deduce that u satisfies the identity (3.3.2). Since φ is an arbitrary test function ; u is a weak solution of (3.1.1). $\square$

Proof of Proposition 3.1.1. Let $T>0$, $R>0$, $X(T):=C([0,T];H^1(\mathbb{R}^n)\cap C^1([0,T];L^2(\mathbb{R}^n))$ and

$$B_R(T)=\{\varphi\in X(T);\ \|\varphi\|_{X(T)}\le 2R\},$$

where $\|\varphi\|_{X(T)}:=\|\nabla\varphi\|_2+\|\varphi_t\|_2$. By Proposition 3.2.5, we define a mapping $\Phi: B_R(T)\to X(T)$ such that $u(t,x)=(\Phi\varphi)(t,x)$ is the unique mild solution to the linear inhomogeneous equation

$$\begin{cases} u_{tt}-\Delta u+a(x)b(t)u_t=f(t,\varphi), & t>0, x\in\mathbb{R}^n,\\ u(0,x)=u_0(x),\ \ u_t(0,x)=u_1(x), & x\in\mathbb{R}^n,\end{cases}$$

moreover

$$\|(u_t,\nabla u)(t)\|_{L^2\times L^2}\le C\,I_0+C\int_0^t\|f(s,\varphi)\|_{L^2}\,ds,$$

where $I_0:=\|u_0\|_{H^1}+\|u_1\|_2$. Using the same calculation as in the proof of Lemma 3.3.3, we have

$$\|f(t,\varphi)\|_2\le C\,T^{1-\gamma}\|\varphi\|_{X(T)}^p.$$

Therefore

$$\|u\|_{X(T)}\le C\,I_0+C\,2^pR^pT^{2-\gamma}.$$

We take R such that

$$R\ge C\,I_0$$

and then choose T sufficiently small so that

$$C\,2^pR^{p-1}T^{2-\gamma}\le 1.$$

It follows that $\|u\|_{X(T)}\le 2R$, whereupon Φ maps $B_R(T)$ into $B_R(T)$.
Next, we show that Φ is a contraction by taking T smaller. Let $\varphi,\overline{\varphi}\in B_R(T)$, $u:=\Phi(\varphi)$, $\overline{u}:=\Phi(\overline{\varphi})$ and $w:=u-\overline{u}$. By Proposition 3.2.5, w is the unique mild solution to the linear inhomogeneous equation

$$\begin{cases} w_{tt}-\Delta w+a(x)b(t)w_t=f(t,\varphi)-f(t,\overline{\varphi}), & t>0, x\in\mathbb{R}^n,\\ w(0,x)=0,\ \ w_t(0,x)=0, & x\in\mathbb{R}^n,\end{cases}$$

moreover

$$\begin{aligned}\|(w_t,\nabla w)(t)\|_{L^2\times L^2} &\le C\int_0^t\|f(s,\varphi)-f(s,\overline{\varphi})\|_{L^2}\,ds\\ &\le C\int_0^t\int_0^s(s-\sigma)^{-\gamma}\||\varphi(\sigma)|^p-|\overline{\varphi}(\sigma)|^p\|_2\,d\sigma\,ds\\ &\le C\int_0^t\int_0^s(s-\sigma)^{-\gamma}\left(\|\varphi^{p-1}(\sigma)\|_{2p/(p-1)}+\|\overline{\varphi}^{p-1}(\sigma)\|_{2p/(p-1)}\right)\|\varphi(\sigma)-\overline{\varphi}(\sigma)\|_{2p}\,d\sigma\,ds\\ &= C\int_0^t\int_0^s(s-\sigma)^{-\gamma}\left(\|\varphi(\sigma)\|_{2p}^{p-1}+\|\overline{\varphi}(\sigma)\|_{2p}^{p-1}\right)\|\varphi(\sigma)-\overline{\varphi}(\sigma)\|_{2p}\,d\sigma\,ds\end{aligned}$$

$$
\begin{aligned}
&\leq C\int_0^t\int_0^s (s-\sigma)^{-\gamma}\left(\|\varphi(\sigma)\|_{H^1}^{p-1}+\|\overline{\varphi}(\sigma)\|_{H^1}^{p-1}\right)\|\varphi(\sigma)-\overline{\varphi}(\sigma)\|_{H^1}\,d\sigma\,ds\\
&\leq C\int_0^t\int_0^s (s-\sigma)^{-\gamma}\left(\|\varphi\|_{X(T)}^{p-1}+\|\overline{\varphi}\|_{X(T)}^{p-1}\right)\|\varphi-\overline{\varphi}\|_{X(T)}\,d\sigma\,ds\\
&\leq C\,2^pR^{p-1}T^{2-\gamma}\|\varphi-\overline{\varphi}\|_{X(T)}\\
&\leq \frac{1}{2}\|\varphi-\overline{\varphi}\|_{X(T)},
\end{aligned}
$$

thanks to Hölder's inequality, the fact that $H^1(\mathbb{R}^n)\hookrightarrow L^{2p}(\mathbb{R}^n)$ and the following inequality

$$
||\varphi|^p-|\overline{\varphi}|^p|\leq C(p)|\varphi-\overline{\varphi}|(|\varphi|^{p-1}+|\overline{\varphi}|^{p-1}); \tag{3.3.3}
$$

T is chosen such that

$$
C\,2^pR^{p-1}T^{2-\gamma}\leq\frac{1}{2}.
$$

This implies that Φ is a contraction. Then, by the Banach fixed point theorem, there exists a mild solution $u\in X(T)$ to problem (3.1.1). Here we have uniqueness of solution in a small time interval but in the presence of the nonlocal term we cannot continue and obtain the uniqueness in a maximal time interval. Therefore, we prove the uniqueness of the solution of (3.1.1) using Gronwall's inequality (cf. [11]). If u and $\overline{u}$ are solutions of (3.1.1), we have

$$
\begin{aligned}
\|u(t)-\overline{u}(t)\|_{H^1} &\leq C\,2^pR^{p-1}\int_0^t\int_0^s (s-\sigma)^{-\gamma}\|u(\sigma)-\overline{u}(\sigma)\|_{2p}\,d\sigma\,ds\\
&= C\,2^pR^{p-1}\int_0^t\int_\sigma^t (s-\sigma)^{-\gamma}\|u(\sigma)-\overline{u}(\sigma)\|_{2p}\,ds\,d\sigma\\
&\leq C\,2^pR^{p-1}\int_0^t (t-\sigma)^{1-\gamma}\|u(\sigma)-\overline{u}(\sigma)\|_{H^1}\,d\sigma.
\end{aligned}
$$

So the uniqueness follows from Gronwall's inequality. Moreover, using the uniqueness of solutions, we conclude the existence of a solution on a maximal interval $[0,T_{\max})$ where

$$
T_{\max}:=\sup\{T>0\,;\ \text{there exist a mild solution } u\in X(T) \text{ to } (3.1.1)\}\leq+\infty.
$$

Finally, if the lifespan $T_{\max}$ is finite, then the weighted energy of the solution blows up at $T_{\max}$:

$$
\lim_{t\to T_{\max}}(\|u(t)\|_{H^1}+\|u_t(t)\|_2)=\infty.
$$

Because, if

$$
\lim_{t\to T_{\max}}(\|u(t)\|_{H^1}+\|u_t(t)\|_2)=:M<\infty,
$$

then there exists a time sequence $\{t_m\}_{m\geq 0}$ tending to $T_{\max}$ as $m\to\infty$ and such that

$$
\sup_{m\in\mathbb{N}}(\|u(t_m)\|_{H^1}+\|u_t(t_m)\|_2)\leq M+1
$$

The argument before shows that there exists $T(M+1)>0$ such that the solution $u(t)$ can be extended on the interval $[t_m,t_m+T(M+1)]$ for any m. By taking m sufficiently large so that $t_m\geq T_{\max}-(1/2)T(M+1)$, the solution $u(t)$ can be extended on $[T_{\max},T_{\max}+(1/2)T(M+1)]$. This contradicts the definition of $T_{\max}$. Thus the proof is completed. □

3.4 Blow-up result

In this section, we present the proof of Theorem 3.1.3.
Proof of Theorem 3.1.3. We assume, on the contrary, that u is a global mild solution of (3.1.1). Let g(t) be the solution of the ordinary differential equation

$$\begin{cases} -g'(t)+(1+t)^{-\beta}g(t)=1, \\ g(0)=g_0, \end{cases}$$

where $g_0 \neq 0$ is a constant. The solution $g(t)$ can be expressed by

$$g(t)=e^{\int_0^t (1+s)^{-\beta}\,ds}\left(g_0-\int_0^t e^{-\int_0^\tau (1+s)^{-\beta}\,ds}\,d\tau\right).$$

By l'Hôpital rule, we have $\lim_{t\to\infty} b(t)g(t)=1$ and $c_1 b(t)^{-1}\leq g(t)\leq c_2 b(t)^{-1}$ due to $\beta\geq -1/3$. We note that $g_0=1$ and $g(t)\equiv 1$ in the case $\beta=0$. By the definition of $g(t)$, we also have $\sup_{t>0}|g'(t)|\leq \sup_{t>0}|b(t)g(t)-1|<\infty$. As u is a mild solution of (3.1.1), from Lemma 3.3.3 we have

$$\begin{aligned}
&\Gamma(\delta)\int_0^T\int_{\mathbb{R}^n} J^\delta_{0|t}(|u|^p)g(t)\varphi\,dx\,dt+\int_{\mathbb{R}^n} u_1(x)g(0)\varphi(0,x)\,dx\\
&+\int_{\mathbb{R}^n} u_0(x)(a(x)g(0)\varphi(0,x)-(g\varphi)_t(0,x))\,dx\\
&=\int_0^T\int_{\mathbb{R}^n} u(g\varphi)_{tt}\,dx\,dt-\int_0^T\int_{\mathbb{R}^n} u(a(x)b(t)g(t)\varphi)_t\,dx\,dt-\int_0^T\int_{\mathbb{R}^n} u\Delta(g\varphi)\,dx\,dt,
\end{aligned}$$

for all $T\gg 1$ and all function $\varphi\in C^2([0,T]\times\mathbb{R}^n)$ compactly supported in x such that $\varphi(\cdot\,,T)=0$ and $\varphi_t(\cdot\,,T)=0$, where $\delta=1-\gamma$. Then

$$\begin{aligned}
&\Gamma(\delta)\int_0^T\int_{\mathbb{R}^n} J^\delta_{0|t}(|u|^p)g(t)\varphi\,dx\,dt+g_0\int_{\mathbb{R}^n} u_1(x)\varphi(0,x)\,dx\\
&+\int_{\mathbb{R}^n} u_0(x)a(x)\varphi(0,x)\,dx-g_0\int_{\mathbb{R}^n} u_0(x)\varphi_t(0,x)\,dx\\
&=\int_0^T\int_{\mathbb{R}^n} ug(t)\varphi_{tt}\,dx\,dt-\int_0^T\int_{\mathbb{R}^n} u(g'(t)-1)a(x)\varphi_t\,dx\,dt-\int_0^T\int_{\mathbb{R}^n} ug(t)\Delta\varphi\,dx\,dt, \quad (3.4.1)
\end{aligned}$$

where we have used the assumption $\alpha\beta=0$.
Let $\varphi(x,t)=D^\delta_{t|T}\left(\tilde{\varphi}(x,t)\right):=D^\delta_{t|T}\left(\varphi_1^\ell(x)\varphi_2(t)\right)$ with $\varphi_1(x):=\Phi\left(|x|/B\right)$, $\varphi_2(t):=(1-t/T)_+^\eta$, where $D^\delta_{t|T}$ is given by (1.1.1), $\ell,\eta\gg 1$ and $\Phi\in C^\infty(\mathbb{R}_+)$ is a cut-off non-increasing function such that

$$\Phi(r)=\begin{cases} 1, & \text{if } 0\leq r\leq 1,\\ 0, & \text{if } r\geq 2,\end{cases}$$

$0 \leq \Phi \leq 1$ and $|\Phi^{'}(r)| \leq C_1/r$ for all $r > 0$. The constant $B > 0$ in the definition of φ_1 is fixed and will be chosen later. In the following, we set :

$$\Omega(B) = \{x \in \mathbb{R}^n : \ |x| \leq 2B\}, \quad \Delta(B) = \{x \in \mathbb{R}^n : \ B \leq |x| \leq 2B\}.$$

Equality (3.4.1) actually reads

$$\begin{aligned}
&\Gamma(\delta)\int_0^T\int_{\Omega(B)} J^{\delta}_{0|t}(|u|^p)g(t)D^{\delta}_{t|T}\tilde{\varphi}\,dx\,dt\\
&+ g_0\int_{\Omega(B)} u_1(x)D^{\delta}_{t|T}\tilde{\varphi}(0,x)\,dx + \int_{\Omega(B)} u_0(x)(a(x)D^{\delta}_{t|T}\tilde{\varphi}(0,x) - g_0\partial_t D^{\delta}_{t|T}\tilde{\varphi}(0,x))\,dx\\
&= \int_0^T\int_{\Omega(B)} u\ g(t)\partial_t^2 D^{\delta}_{t|T}\tilde{\varphi}\,dx\,dt - \int_0^T\int_{\mathbb{R}^n} u(g'(t)-1)a(x)\partial_t(D^{\delta}_{t|T}\tilde{\varphi})\,dx\,dt\\
&-\int_0^T\int_{\mathbb{R}^n} ug(t)\Delta D^{\delta}_{t|T}\tilde{\varphi}\,dx\,dt.
\end{aligned}$$

As $c_1 b(t)^{-1} \leq g(t) \leq c_2 b(t)^{-1}$ and $b(s) > b(t)$ for $0 \leq s \leq t$, we have

$$g(t) \geq \frac{c_1}{b(t)} = \frac{c_1c_2}{c_2b(t)} \geq \frac{c_1g(s)b(s)}{c_2b(t)} \geq \frac{c_1}{c_2}g(s),$$

then

$$\begin{aligned}
&\Gamma(\delta)\int_0^T\int_{\Omega(B)} J^{\delta}_{0|t}(|u|^p g)D^{\delta}_{t|T}\tilde{\varphi}\,dx\,dt\\
&+ g_0\int_{\Omega(B)} u_1(x)D^{\delta}_{t|T}\tilde{\varphi}(0,x)\,dx + \int_{\Omega(B)} u_0(x)(a(x)D^{\delta}_{t|T}\tilde{\varphi}(0,x) - g_0\partial_t D^{\delta}_{t|T}\tilde{\varphi}(0,x))\,dx\\
&\leq \frac{c_2}{c_1}\Gamma(\delta)\int_0^T\int_{\Omega(B)} J^{\delta}_{0|t}(|u|^p)g(t)D^{\delta}_{t|T}\tilde{\varphi}\,dx\,dt\\
&+ g_0\int_{\Omega(B)} u_1(x)D^{\delta}_{t|T}\tilde{\varphi}(0,x)\,dx + \int_{\Omega(B)} u_0(x)(a(x)D^{\delta}_{t|T}\tilde{\varphi}(0,x) - g_0\partial_t D^{\delta}_{t|T}\tilde{\varphi}(0,x))\,dx\\
&= \int_0^T\int_{\Omega(B)} u\ g(t)\partial_t^2 D^{\delta}_{t|T}\tilde{\varphi}\,dx\,dt - \int_0^T\int_{\mathbb{R}^n} u(g'(t)-1)a(x)\partial_t(D^{\delta}_{t|T}\tilde{\varphi})\,dx\,dt\\
&-\int_0^T\int_{\mathbb{R}^n} ug(t)\Delta D^{\delta}_{t|T}\tilde{\varphi}\,dx\,dt.
\end{aligned}$$

From (1.1.3), (1.1.4) and (1.1.7), we conclude that

$$\begin{aligned}
&\int_0^T\int_{\Omega(B)} D^{\delta}_{0|t}J^{\delta}_{0|t}(|u|^p g)\tilde{\varphi}\,dx\,dt + C\,T^{-\delta-1}\int_{\Omega(B)}\left[T(u_1(x)+a(x)u_0(x))+u_0(x)\right]\varphi_1^{\ell}(x)\,dx\\
&\leq C\int_0^T\int_{\Omega(B)} u\ g(t)D^{2+\delta}_{t|T}\tilde{\varphi}\,dx\,dt + C\int_0^T\int_{\mathbb{R}^n} u(g'(t)-1)a(x)D^{1+\delta}_{t|T}\tilde{\varphi}\,dx\,dt\\
&-C\int_0^T\int_{\mathbb{R}^n} ug(t)\Delta\varphi_1^{\ell}(x)D^{\delta}_{t|T}\varphi_2(t)\,dx\,dt,
\end{aligned}$$

where $D^{\delta}_{0|t}$ is defined in (1.1.1). Moreover, using (1.1.5) and the fact that (**??**) implies

$$C\int_{\Omega(B)} u_0(x)\varphi_1^{\ell}(x)\,dx \geq \int_{\Omega(B)} a(x)u_0(x)\varphi_1^{\ell}(x)\,dx \geq 0, \qquad \int_{\Omega(B)} u_1(x)\varphi_1^{\ell}(x)\,dx \geq 0,$$

it follows that

$$\begin{aligned}\int_0^T\int_{\Omega(B)} |u|^p g(t)\tilde{\varphi}\,dx\,dt &\leq C\int_0^T\int_{\Omega(B)} |u|g(t)\varphi_1^{\ell} D^{2+\delta}_{t|T}\varphi_2\,dx\,dt + C\int_0^T\int_{\Omega(B)} |u|a(x)\varphi_1^{\ell} D^{1+\delta}_{t|T}\varphi_2\,dx\,dt\\ &\quad + C\int_0^T\int_{\Delta(B)} |u|g(t)\varphi_1^{\ell-2}(|\Delta\varphi_1| + |\nabla\varphi_1|^2)D^{\delta}_{t|T}\varphi_2\,dx\,dt\\ &=: I_1 + I_2 + I_3, \end{aligned} \tag{3.4.2}$$

where we have used the formula $\Delta(\varphi_1^{\ell}) = \ell\varphi_1^{\ell-1}\Delta\varphi_1 + \ell(\ell-1)\varphi_1^{\ell-2}|\nabla\varphi_1|^2$, and the fact that $\varphi_1 \leq 1$ and $|g'(t) - 1| \leq C$. Next by applying Young's inequality, we obtain

$$\begin{aligned} I_1 &= C\int_0^T\int_{\Omega(B)} |u|g^{1/p}(t)\tilde{\varphi}^{1/p}\tilde{\varphi}^{-1/p}g^{1-1/p}(t)\varphi_1^{\ell}D^{2+\delta}_{t|T}\varphi_2\,dx\,dt\\ &\leq \frac{1}{3p}\int_0^T\int_{\Omega(B)} |u|^p g(t)\tilde{\varphi}\,dx\,dt + C\int_0^T\int_{\Omega(B)} g(t)\varphi_1^{\ell}\varphi_2^{-1/(p-1)}(D^{2+\delta}_{t|T}\varphi_2)^{p'}\,dx\,dt, \end{aligned} \tag{3.4.3}$$

where $p' = p/(p-1)$. Similarly,

$$I_2 \leq \frac{1}{3p}\int_0^T\int_{\Omega(B)} |u|^p g(t)\tilde{\varphi}\,dx\,dt + C\int_0^T\int_{\Omega(B)} \varphi_1^{\ell} g^{-1/(p-1)}(t)\varphi_2^{-1/(p-1)}(a(x))^{p'}(D^{1+\delta}_{t|T}\varphi_2)^{p'}\,dx\,dt \tag{3.4.4}$$

and

$$I_3 \leq \frac{1}{3p}\int_0^T\int_{\Omega(B)} |u|^p g(t)\tilde{\varphi}\,dx\,dt + C\int_0^T\int_{\Omega(B)} g(t)\varphi_1^{\ell-2p'}\varphi_2^{-1/(p-1)}(|\Delta\varphi_1|^{p'} + |\nabla\varphi_1|^{2p'})(D^{\delta}_{t|T}\varphi_2)^{p'}\,dx\,dt. \tag{3.4.5}$$

Using (3.4.3), (3.4.4) and (3.4.5), it follows from (3.4.2) that

$$\begin{aligned}\int_0^T\int_{\Omega(B)} |u|^p g(t)\tilde{\varphi} &\leq C\int_0^T\int_{\Omega(B)} g(t)\varphi_1^{\ell}\varphi_2^{-1/(p-1)}(D^{2+\delta}_{t|T}\varphi_2)^{p'}\\ &+ C\int_0^T\int_{\Omega(B)} \varphi_1^{\ell} g^{-1/(p-1)}(t)\varphi_2^{-1/(p-1)}(a(x))^{p'}(D^{1+\delta}_{t|T}\varphi_2)^{p'}\\ &+ C\int_0^T\int_{\Omega(B)} g(t)\varphi_1^{\ell-2p'}\varphi_2^{-1/(p-1)}(|\Delta\varphi_1|^{p'} + |\nabla\varphi_1|^{2p'})(D^{\delta}_{t|T}\varphi_2)^{p'}. \end{aligned} \tag{3.4.6}$$

At this stage, to prove i), we have to distinguish 2 cases.

- <u>Case $1 < p < p_c$</u> : in this case, we take $B = T^{\frac{1}{2-\alpha}}$. As $B < T$, and $g^{-1}(t) \leq C$, we have

$$\int_0^T\int_{\Omega(B)} |u|^p g(t)\tilde{\varphi}\,dx\,dt \leq C\int_0^T\int_{\Omega(T)} g(t)\varphi_1^{\ell}\varphi_2^{-1/(p-1)}(D^{2+\delta}_{t|T}\varphi_2)^{p'}\,dx\,dt$$

$$\begin{aligned}
&+C\int_0^T\int_{\Omega(T^{1/(2-\alpha)})}\varphi_1^{\ell}\varphi_2^{-1/(p-1)}(a(x))^{p'}(D_{t|T}^{1+\delta}\varphi_2)^{p'}\,dx\,dt\\
&+C\int_0^T\int_{\Omega(T)}g(t)\varphi_1^{\ell-2p'}\varphi_2^{-1/(p-1)}(|\Delta\varphi_1|^{p'}+|\nabla\varphi_1|^{2p'})(D_{t|T}^{\delta}\varphi_2)^{p'}\,dx\,dt\\
=:\quad &J_1+J_2+J_3. \qquad (3.4.7)
\end{aligned}$$

So, using the change of variables : $(s=T^{-1}t,\, y=T^{-1}x)$ in J_1 and J_3, and $(s=T^{-1}t,\, y=T^{-\frac{1}{2-\alpha}}x)$ in J_2 we get from (1.1.6) that

$$J_1\le C\,T^{\beta-(2+\delta)p'+1+n}\le C\,T^{1-\alpha-(2+\delta)p'+1+n}=C\,T^{-((2+\delta)p'-(2-\alpha)-n)},$$

$$J_3\le C\,T^{\beta-2p'-\delta p'+1+n}\le C\,T^{1-\alpha-(2+\delta)p'+1+n}=C\,T^{-((2+\delta)p'-(2-\alpha)-n)},$$

and

$$J_2\le C\begin{cases}T^{-a_1}, & \text{if } n>\alpha p',\\ T^{-a_2}\ln T, & \text{if } n=\alpha p',\\ T^{-a_2}, & \text{if } n<\alpha p',\end{cases}$$

where

$$a_1:=\frac{1}{2-\alpha}((2+\delta(2-\alpha))p'-(2-\alpha)-n),$$

and

$$a_2:=\frac{1}{2-\alpha}((2+\delta(2-\alpha))p'-2).$$

Therefore, we conclude from (3.4.7) that

$$\int_0^T\int_{\Omega(T^{\frac{1}{2-\alpha}})}|u|^p g(t)\tilde{\varphi}\,dx\,dt\le C\begin{cases}T^{-a_1}, & \text{if } n>\alpha p',\\ T^{-a_1}+T^{-a_2}\ln T, & \text{if } n=\alpha p',\\ T^{-a_1}+T^{-a_2}, & \text{if } n<\alpha p'.\end{cases} \qquad (3.4.8)$$

Letting $T\to\infty$ in (3.4.8), thanks to $p<p_c$ and the Lebesgue dominated convergence theorem, it yields that

$$\int_0^\infty\int_{\mathbb{R}^n}|u|^p g(t)\,dx\,dt=0,$$

which implies $u(x,t)=0$ for all t and a.e. x. This contradicts our assumption (**??**).

• Case $p=p_c$: let $B=R^{-\frac{1}{2-\alpha}}T^{\frac{1}{2-\alpha}}$, where $1\ll R<T$ is such that T and R do not go simultaneously to ∞. Moreover, from the first case and the fact that $p=p_c$, there exists a positive constant D independent of T such that

$$\int_0^\infty\int_{\mathbb{R}^n}|u|^p g(t)\,dx\,dt\le D,$$

which implies that

$$\int_0^T\int_{\Delta(R^{-\frac{1}{2-\alpha}}T^{\frac{1}{2-\alpha}})}|u|^p g(t)\tilde{\varphi}\,dx\,dt\to 0\quad\text{as}\quad T\to\infty. \qquad (3.4.9)$$

On the other hand, using Hölder's inequality instead of Young's one, we estimate the integral I_3 in (3.4.2) as follows :

$$\begin{aligned} I_3 &\leq C\left(\int_0^T\int_{\Delta(R^{-\frac{1}{2-\alpha}}T^{\frac{1}{2-\alpha}})}|u|^p g(t)\tilde{\varphi}\right)^{1/p} \\ &\quad\times\left(\int_0^T\int_{\Omega(R^{-1}T)} g(t)\varphi_1^{\ell-2p'}\varphi_2^{-1/(p-1)}(|\Delta\varphi_1|^{p'}+|\nabla\varphi_1|^{2p'})(D_{t|T}^{\alpha}\varphi_2)^{p'}\,dx\,dt\right)^{1/p'} \end{aligned} \tag{3.4.10}$$

Similarly to the last case, substituting (3.4.3), (3.4.4) and (3.4.10) into (3.4.2), we get

$$\begin{aligned} \int_0^T\int_{\Omega(B)}|u|^p g(t)\tilde{\varphi}\,dx\,dt &\leq C\int_0^T\int_{\Omega(R^{-1}T)} g(t)\varphi_1^{\ell}\varphi_2^{-1/(p-1)}(D_{t|T}^{2+\delta}\varphi_2)^{p'}\,dx\,dt \\ &\quad + C\int_0^T\int_{\Omega(R^{-\frac{1}{2-\alpha}}T^{\frac{1}{2-\alpha}})}\varphi_1^{\ell}\varphi_2^{-1/(p-1)}(a(x))^{p'}(D_{t|T}^{1+\delta}\varphi_2)^{p'}\,dx\,dt \\ &\quad + C\left(\int_0^T\int_{\Delta(R^{-\frac{1}{2-\alpha}}T^{\frac{1}{2-\alpha}})}|u|^p g(t)\tilde{\varphi}\right)^{1/p} \\ &\quad\times\left(\int_0^T\int_{\Omega(R^{-1}T)} g(t)\varphi_1^{\ell-2p'}\varphi_2^{-1/(p-1)}(|\Delta\varphi_1|^{p'}+|\nabla\varphi_1|^{2p'})(D_{t|T}^{\alpha}\varphi_2)^{p'}\,dx\,dt\right)^{1/p'} \\ &=: K_1+K_2+K_3. \end{aligned}$$

Taking account of $p=p_c$ and the scaled variables ($s=T^{-1}t$, $y=RT^{-1}x$) in K_1 and K_3, and ($s=T^{-1}t$, $y=R^{\frac{1}{2-\alpha}}T^{-\frac{1}{2-\alpha}}x$) in K_2 we conclude that

$$\int_0^T\int_{\Omega(R^{-\frac{1}{2-\alpha}}T^{\frac{1}{2-\alpha}})}|u|^p g(t)\tilde{\varphi}\,dx\,dt \leq C\,R^{-n}+C\,R^{-\frac{n-\alpha p'}{2-\alpha}}+C\,R^{2-\frac{n}{p'}}\left(\int_0^T\int_{\Delta(R^{-\frac{1}{2-\alpha}}T^{\frac{1}{2-\alpha}})}|u|^p g(t)\tilde{\varphi}\right)^{1/p}.$$

Letting $T\to\infty$, using (3.4.9), we get

$$\int_0^\infty\int_{\mathbb{R}^n}|u|^p g(t)\,dx\,dt \leq C\,R^{-n}+C\,R^{-\frac{n-\alpha p'}{2-\alpha}},$$

which implies a contradiction, when $R\to\infty$, with (**??**) where we have used the fact that $p=p_c>\frac{n}{n-\alpha}$. This completes the proof of Theorem 3.1.3, i).

For the proof of ii), we have two possibilities :

- $\gamma<(n+2(\alpha-2))/n$: let $B=R$ with the same R introduced in the case $p=p_c$. Then, taking the scaled variables $s=T^{-1}t$, $y=R^{-1}x$, it follows from (3.4.6) that

$$\int_0^T\int_{\Omega(R)}|u|^p g(t)\tilde{\varphi}\,dx\,dt \leq C\begin{cases} R^n\,T^{-(2+\delta)p'+1}+R^{n-\alpha p'}\,T^{-(1+\delta)p'+1}+R^{n-2p'}\,T^{-\delta p'+1}, & \text{if } n>\alpha p' \\ R^n\,T^{-(2+\delta)p'+1}+\ln R\,T^{-(1+\delta)p'+1}+R^{n-2p'}\,T^{-\delta p'+1}, & \text{if } n=\alpha p' \\ R^n\,T^{-(2+\delta)p'+1}+T^{-(1+\delta)p'+1}+R^{n-2p'}\,T^{-\delta p'+1}, & \text{if } n<\alpha p'. \end{cases}$$

As $\gamma < (n+2(\alpha-2))/n$ implies $p \leq n/n-2 < 1/\gamma$, we get a contradiction with (**??**) by letting the following limits : first $T \to \infty$, next $R \to \infty$.

• $\underline{\gamma = (n+2(\alpha-2))/n}$: we have $p \leq n/(n-2) = p_c$. Using i), we get the contradiction. This completes the proof of Theorem 3.1.3, *ii*). $\square$

Chapitre 4

On the Cauchy problem for an evolution p-Laplacian equation with a nonlinear memory

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Abstract

We consider the Cauchy problem in $\mathbb{R}^n$, $n \geq 1$, for the evolution p-Laplacian equation with a nonlinear memory. We obtain the local existence of solutions. We prove the nonexistence of nonnegative global solutions.

Keywords : Cauchy problem, p-Laplacian equation, Local existence, Global nonexistence.
MSC : 35K65; 35A01;

4.1 Introduction

This paper is concerned with the Cauchy problem for the following evolution p-Laplacian equation

$$\begin{cases} u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \displaystyle\int_0^t (t-s)^{-\gamma}|u(s)|^{q-1}u(s)\,ds, & t>0, x\in\mathbb{R}^n, \\ u(0,x) = u_0(x), & x\in\mathbb{R}^n, \end{cases} \tag{4.1.1}$$

where $p>2$, $q>1$, $0<\gamma<1$, and $u_0 \in L^\infty(\mathbb{R}^n)$.

The study of blow-up for nonlinear parabolic equations originated from Fujita [24]. He studied the following Cauchy problem of the semilinear heat equation

$$\begin{cases} u_t - \Delta u = u^p, u \geq 0 & t>0, x\in\mathbb{R}^n, \\ u(0,x) = u_0(x), & x\in\mathbb{R}^n, \end{cases} \tag{4.1.2}$$

where $p > 1$, and obtains the following results :

a) If $p < 1 + 2/n$, then every nontrivial solution of (4.1.2) blows up in finite time.

b) If $p > 1 + 2/n$ and $u_0(x) \leq \delta e^{-|x|^2}$, $(0 < \delta \ll 1)$, then (4.1.2) admits a global solution.

In the case $p = 1 + 2/n$, it is shown by the author of [27] for $n = 1, 2$ and by the authors of [40] for $n \geq 1$ that (4.1.2) has no global solution $u(x,t)$ satisfying $\|u(\cdot\,,t)\|_\infty < \infty$ for $t \geq 0$. The author of [68] proves that if $p = 1 + 2/n$, (4.1.2) has no global solution $u(x,t)$ satisfying $\|u(\cdot\,,t)\|_q < \infty$ for $t > 0$ and some $q \in [1, +\infty)$. The value $p_c = 1 + 2/n$ is called the critical exponent of (4.1.2). It plays an important role in the behavior of the solution to (4.1.2).

In the sixty years, there have been a number of extensions of Fujita's results in many directions. Recently, the authors of [] extended Fujita's results to the heat equation with nonlinear memory term

$$\begin{cases} u_t - \Delta u = \displaystyle\int_0^t (t-s)^{-\gamma} u(s)^p \, ds, & t > 0, x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{4.1.3}$$

where $p > 1$, $0 < \gamma < 1$, and $u_0 \in C_0(\mathbb{R}^n)$. In this case, the value of the critical Fujita exponent is $p_c = \max\{\frac{1}{\gamma}, 1 + 2(2-\gamma)/(N-2+2\gamma)_+\}$. Moreover, the nonlinear nonlocal term can be considered as an approximation of the nonlinear term in the semilinear heat equation (4.1.3) since the limit

$$\lim_{\gamma \to 1} s_+^{-\gamma} = \Gamma(1-\gamma)\delta(s)$$

exists in distribution sense, where Γ is the Euler gamma function.

For the evolution p-Laplacian equation

$$\begin{cases} u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = u^q, u \geq 0 & t > 0, x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{4.1.4}$$

where $p > 2$, $q > 1$, the author of [71] discussed the Cauchy problem and obtained the critical exponent $q_c = p - 1 + p/n$ provided $p > 2n/(n+1)$. Furthermore, the author demonstrated that if $\max\{1, p-1\} < q < q_c$, then the Cauchy problem has no global solution ; whereas if $q > q_c$ and $u_0(x)$ is small enough, then the Cauchy problem admits a global positive solution. The authors of [2, 49] discussed more general quasi-linear parabolic equations and the doubly singular parabolic equations respectively, and obtain similar results.

Our goal is to study the local existence of a solution of equation (4.1.1) as well as the nonexistence of the global solution. Our main result is

Theorem 4.1.1 (Global nonexistence)
If $q \leq q_c$, then problem (4.1.1) has no nontrivial weak solutions.

This paper is organized as follows : Section 4.2 contains the local existence theorem. In Section 4.3, the global nonexistence result (Theorem 4.1.1) will be proved.

Throughout this paper, positive constants will be denoted by C and will change from line to line.

4.2 Local existence

Using the Riemann-Liouville integral operator ; therefore, (4.1.1) takes the form

$$u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \Gamma(\delta) J^{\delta}_{0|t}\left(|u|^p\right)(t), \tag{4.2.1}$$

where $\delta = 1-\gamma$. Let us present our main results.

First, we give the definition of the

Definition 4.2.1 (Weak solution) *A measurable function $u(x,t)$ defined in $S_T = \mathbb{R}^N \times (0,T]$ is called a weak solution of (4.1.1) if for every bounded open set Ω with smooth boundary $\partial\Omega$,*

$$u \in C([0,T] : L^1(\Omega)) \cap L^p_{loc}(0,T : W^1_p(\Omega)) \cap L^\infty_{loc}(S_T),$$

and

$$\begin{aligned}
&\int_\Omega u(x,t)\phi(x,t)\,dx + \int_{t_0}^t \int_\Omega [-u\phi_t + |\nabla u|^{p-2}\nabla u \cdot \nabla\phi]\,dx\,d\tau \\
&= \Gamma(\delta)\int_{t_0}^t \int_\Omega J^{\delta}_{0|\tau}(|u|^{q-1}u)\varphi\,dxd\tau + \int_\Omega u(x,t_0)\phi(x,t_0)\,dx
\end{aligned} \tag{4.2.2}$$

for all $0 \le t_0 < t \le T$ and $\phi \in C^1(\overline{\Omega}\times[0,T])$, $\phi(\cdot,T) = 0$ near $\partial\Omega$, where $\delta = 1-\gamma$. Moreover

$$\lim_{t\mapsto 0}\int_{B_r} |u(x,t) - u_0(x)|\,dx = 0, \quad \forall r > 0. \tag{4.2.3}$$

We use $\nu(a_1, a_2, \cdots, a_n)$ to denote positive constants depending only on $a_1, a_2, \cdots, a_n$. Let

$$\operatorname{sgn}_\eta s = \begin{cases} 1 & \text{if} \quad s > \eta, \\ \dfrac{s}{\eta} & \text{if} \quad -\eta \le s \le \eta, \\ -1 & \text{if} \quad s < -\eta. \end{cases}$$

For $f \in L_{loc}(\mathbb{R}^N)$, we define

$$|||f|||_h = \sup_{x\in\mathbb{R}^N}\left(\frac{1}{|B_1(x)|}\int_{B_1(x)} |f(y)|^h\,dy\right)^{\frac{1}{h}}. \tag{4.2.4}$$

Theorem 4.2.2 (Local existence) *Let $|||u_0|||_h < \infty$ where*

$$h = 1 \quad \text{if} \quad 1 \le q < p-1+\frac{p}{N} \quad \text{and} \quad h > \left(\frac{N}{p}\right)(q-p+1) \quad \text{if} \quad q \ge p-1+\frac{p}{N}. \tag{4.2.5}$$

Then there exist a constant $\nu = \nu(N,p,q) \ge 1$ and a positive constant T_0 defined by

$$T_0 + T_0|||u_0|||_h^{p-2} + T_0^{1-\frac{N(q-p+1)}{ph}}|||u_0|||_h^{q-1} = \nu^{-1} \tag{4.2.6}$$

such that there exists a weak solution u to (4.1.1) in the strip S_T satisfying, for all $0 < t < T_0$,

$$|||u(\cdot,t)|||_h \le \nu |||u_0|||_h, \tag{4.2.7}$$

$$|u(x,t)| \le \nu t^{-\frac{N}{k_h}+\frac{\gamma-1}{q-1}} |||u_0|||_h^{\frac{ph}{k_h}}, \tag{4.2.8}$$

where $k_h = N(p-2) + ph$.

Proof. The proof of the Theorem 4.2.2 is based on the following lemmas.

Lemma 4.2.3 *Let u be any locally bounded continuous weak solution of (4.1.1) in S_T for some $0 < T < \infty$. Then for fixed $h \ge 1$ there exists a constant ν depending only on N, p, q, h such that for every ball $B_{2\rho}(x_0)$ and for all $0 < t < T$,*

$$||u(\cdot,\tau)||_{\infty,B_{2\rho}(x_0)}^{p-2} \rho^{-p} + \tau^{1-\gamma} \sup_{x\in B_{2\rho}(x_0)} |u|^{q-1} \le \tau^{-1}, \quad \tau \in (0,t). \tag{4.2.9}$$

Furthermore, the following estimate holds

$$||u(\cdot,t)||_{\infty,B_\rho(x_0)} \le \nu t^{-\frac{N+p}{k_h}} \left(\int_0^t \int_{B_{2\rho}(x_0)} |u|^h \, dx \, d\tau \right)^{\frac{p}{k_h}}, \tag{4.2.10}$$

where $k_h = N(p-2) + ph$.

Lemma 4.2.4 *Let the assumptions of Lemma 4.2.3 hold and set*

$$G(t) = \sup_{0<\tau<t} \left(\int_{B_2(x_0)} |u(x,\tau)|^h \, dx \right)^{\frac{1}{h}} < \infty. \tag{4.2.11}$$

Then there exists a constant $\nu = \nu(N,p,q)$ such that for every ball $B_2(x_0)$, $0 < t < T$, u satisfies (4.2.9)

$$\int_0^t \int_{B_1(x_0)} |\nabla u|^{p-1} \, dx \, d\tau \le \nu t^{\frac{h}{k_h}} G(t)^{1+\frac{h(p-2)}{k_h}}, \quad k_h = N(p-2) + ph. \tag{4.2.12}$$

We postpone the proofs of Lemma 4.2.3 and Lemma 4.2.4 at the end of this section.

Define a sequence of functions $\{f_n\}$ satisfying

$$f_n \in C^\infty(\mathbb{R}^N), \quad f_n(r) = r \quad \text{for } r \in [-n,n], \quad f_n(r) = n+1 \quad \text{for } |r| \ge n+1 \quad \text{and } |f'| \le \nu.$$

Consider the family of approximating problem

$$\begin{cases} (u_n)_t = \mathrm{div}(|\nabla u_n|^{p-2}\nabla u_n) + C\ J^\delta_{0|t}(|f_n(u_n)|^{q-1} f_n(u_n)) & \text{in} \quad Q_n, \\ Q_n = B_n \times \mathbb{R}^+, \quad B_n = \{|y| < n\}, \\ u_n(y,t) = 0, \quad \text{for } |y| = n, \\ u_n(y,0) = u_{0n}(y), \end{cases} \tag{4.2.13}$$

where $u_{0n} \in C_0^\infty(B_n)$ satisfies

$$\lim_{n\mapsto\infty} \int_{B_\rho} |u_{0n} - u_0|\, dy = 0, \quad \forall \rho > 0,$$

and

$$|||u_{0n}|||_h \leq |||u_0|||_h.$$

By results of [16, 18] and [17], there exists a solution $u_n \in C(\overline{S_T}) \cap L^\infty(S_T)$, $\nabla u_n \in C^{\beta,\frac{\beta}{2}}(\overline{\Omega} \times (\epsilon, T))$, $(u_n)_t \in L^2(0, T : L^2_{loc}(\mathbb{R}^N))$ to (4.2.13), where Ω can be any bounded open set, $\epsilon, T > 0$ and some $\beta \in (0, 1)$. Therefore for all $t \in \mathbb{R}$

$$\sup_{0<\tau<t} \sup_{y\in\mathbb{R}^N} \{|u_n|^{p-2}(y,\tau) + \tau^{1-\gamma}|u_n|^{q-1}(y,\tau)\} \leq C(n) \tag{4.2.14}$$

for a qualitative constant $C(n)$ depending on the Theorem 4.2.2 will follow by a standard limiting process via the compactness results of [16] and [18] whence we show estimates (4.2.7)-(4.2.8) with u and u_0 replaced by u_n and u_{n0} with constants independent of n. To prove these estimates we will work with (4.2.13) and drop the subscript n.

Let $B_\rho(x)$ denote the ball with center x and radius ρ. Let $\bar{t}$ be the largest time satisfying that for all $t \in (0, \bar{t})$

$$\sup_{x\in\mathbb{R}^N} \left\{ ||u(\cdot,t)||^{p-2}_{\infty,B_2(x)} + t^{1-\gamma} \sup_{y\in B_2(x)} |u|^{q-1}(y,t) \right\} \leq t^{-1}. \tag{4.2.15}$$

Thus by Lemma 4.2.3, there exists a constant $\nu = \nu(N.p, q)$ independent of n such that

$$||u(\cdot,\tau)||_{\infty,B_1(x)} \leq \nu t^{-\frac{N+p}{k_h}} \left(\int_0^t \int_{B_2(x)} |u|^h \, dx\, d\tau \right)^{\frac{p}{k_h}}, \tag{4.2.16}$$

where $k_h = N(p-2) + ph$, for all $0 < t < \bar{t}$.

Set

$$\psi(t) = \sup_{0<\tau<t} |||u(\cdot,\tau)|||_h^h$$

and observe that $\psi(t)$ is finite.

Next we assume (4.2.5) holds. It follows from (4.2.16) that for all $0 < t < \bar{t}$

$$\begin{aligned} t \sup_{x\in\mathbb{R}^N} |u|^{p-2} &\leq \nu t^{1-\frac{N(p-2)}{k_h}} \psi^{\frac{p(p-2)}{k_h}}(t), \\ t^{2-\gamma} \sup_{x\in\mathbb{R}^N} |u|^{q-1} &\leq \nu t^{1-\frac{N(q-1)}{k_h}} \psi^{\frac{p(q-1)}{k_h}}(t). \end{aligned} \tag{4.2.17}$$

Also for $\delta > 0$ to be chosen, we define

$$t^* = \sup\{t > 0; t^h \psi^{p-2}(t) + t^{\frac{1}{p}(k_h - N(q-1))}\psi^{q-1}(t) \leq \delta\}. \tag{4.2.18}$$

Notice that $k_h - N(q-1) > 0$. Therefore for all $0 < t < \min\{\bar{t}, t^*\}$,

$$t \sup_{x\in\mathbb{R}^N} |u|^{p-2}(x,t) + t^{2-\gamma} \sup_{x\in\mathbb{R}^N} |u|^{q-1}(x,t) \leq \nu\delta^{\frac{p}{k_h}}.$$

If follows that $\delta = \delta(p, q, N)$ can be chosen small enough such that $t^* \le \bar{t}$. Let $\xi(x)$ be nonnegative smooth cutoff function in $B_2(x)$ such that $\xi = 1$ on $B_1(x)$, $|\nabla \xi| \le \nu$. We use $(u_\pm)^{h-1}\xi^p$ as a testing function in (4.2.13). If $h > 1$, we get

$$\begin{aligned}\int_0^t \int_{B_2(x)} u_t u^{h-1}\xi^p(y)\,dy\,d\tau &= \int_0^t \int_{B_2(x)} |\nabla u|^{p-2}\nabla u \nabla (u^{h-1}\xi^p(y))\,dy\,d\tau \\ &+ \Gamma(\delta)\int_0^t \int_{B_2(x)} J^\delta_{0|t}(|u|^{q-1}u)u^{h-1}\xi^p(y)\,dy\,d\tau.\end{aligned}$$

Then

$$\begin{aligned}&\int_{B_2(x)} u^h(y,t)\xi^p\,dy + h(h-1)\int_0^t \int_{B_2(x)} |\nabla u|^p u^{h-2}\xi^p(y)\,dy\,d\tau \\ &\le \int_{B_2(x)} u_0^h(y)\xi^p\,dy + ph\int_0^t \int_{B_2(x)} |\nabla u|^{p-1}u^{h-1}|\nabla \xi|\xi^{p-1}(y)\,dy\,d\tau \\ &+\Gamma(\delta)\int_0^t \int_{B_2(x)} J^\delta_{0|t}(|u|^{q-1}u)u^{h-1}\xi^p(y)\,dy\,d\tau.\end{aligned}$$

By the Hölder inequality, we have

$$\begin{aligned}&\int_0^t \int_{B_2(x)} |\nabla u|^{p-1}u^{h-1}|\nabla \xi|\xi^{p-1}(y)\,dy\,d\tau \\ &\le \int_0^t \int_{B_2(x)} |\nabla u|^{p-1}u^{h-1}\,dy\,d\tau \\ &\le \int_0^t \int_{B_2(x)} u^{h-1-\frac{(h-2)(p-1)}{p}}|\nabla u|^{p-1}u^{\frac{(h-2)(p-1)}{p}}\,dy\,d\tau \\ &\le \int_0^t \int_{B_2(x)} u^{p(h-1)-(h-2)(p-1)}\,dy\,d\tau + \int_0^t \int_{B_2(x)} |\nabla u|^p u^{h-2}\,dy\,d\tau \\ &= \int_0^t \int_{B_2(x)} u^{p+h-2}\,dy\,d\tau + \int_0^t \int_{B_2(x)} |\nabla u|^p u^{h-2}\,dy\,d\tau.\end{aligned}$$

Hence

$$\begin{aligned}&\int_{B_2(x)} u^h(y,t)\,dy + C\int_0^t \int_{B_2(x)} |\nabla u|^p u^{h-2}\xi^p(y)\,dy\,d\tau \\ &\le \int_{B_2(x)} u^h(y,t)\,dy \qquad (4.2.19)\\ &\le \int_{B_2(x)} u_0^h(y)\,dy + \int_0^t \int_{B_2(x)} u^{p+h-2}\,dy\,d\tau + \Gamma(\delta)\int_0^t \int_{B_2(x)} J^\delta_{0|t}(|u|^{q-1}u)u^{h-1}\xi^p(y)\,dy\,d\tau,\end{aligned}$$

where C is a constant depending on p and h.

On another hand

$$
\begin{aligned}
&\Gamma(\delta)\int_0^t\int_{B_2(x)} J^{\delta}_{0|t}(|u|^{q-1}u)u^{h-1}\xi^p\,dy\,d\tau \qquad (4.2.20)\\
&\le \Gamma(\delta)\int_0^t\int_{B_2(x)}\left(\int_0^s (s-\sigma)^{-\gamma}u^q(\sigma)\,d\sigma\right)u^{h-1}\,dy\,d\tau\\
&\le \Gamma(\delta)\int_0^t ||\int_0^s (s-\sigma)^{-\gamma}u^q(\sigma)\,d\sigma||^h_{L^h_x(B_2(x))}\times ||u^{h-1}||^{\frac{h}{h-1}}_{L_x^{\frac{h}{h-1}}(B_2(x))}\,dy\,d\tau\\
&\le \Gamma(\delta)\left(\int_0^t ||\int_0^s (s-\sigma)^{-\gamma}u^q(\sigma)\,d\sigma||^h_{L^h_x(B_2(x))}\,dy\,d\tau\right)^{\frac{1}{h}}\times\left(\int_0^t ||u^{h-1}||^{\frac{h}{h-1}}_{L_x^{\frac{h}{h-1}}(B_2(x))}\,dy\,d\tau\right)^{\frac{h-1}{h}}\\
&\le \Gamma(\delta)\,c\,t^{1-\gamma}||u(\cdot,t)||^{q-1}_{\infty,B_{2\rho}(x)}\left(\int_0^t\int_{B_2(x)}u^h(y,\tau)\,dy\,d\tau\right)^{\frac{1}{h}}\times\left(\int_0^t\int_{B_2(x)}u^h(y,\tau)\,dy\,d\tau\right)^{\frac{h-1}{h}}\\
&= \Gamma(\delta)\,c\,t^{1-\gamma}||u(\cdot,t)||^{q-1}_{\infty,B_{2\rho}(x)}\int_0^t\int_{B_2(x)}u^h(y.\tau)\,dy\,d\tau.
\end{aligned}
$$

Combining (4.2.17), (4.2.19) and (4.2.20), we get

$$
\begin{aligned}
&\int_{B_2(x)} u^h(y,t)\,dy\\
&\le \int_{B_2(x)} u_0^h(y)\,dy+\nu\left\{\sup_{0<\tau<t}\sup_{x\in\mathbb{R}^N}\int_{B_2(x)}u^h(y,t)\,dy\right\}\\
&\quad\times\left\{\int_0^t \tau^{-\frac{N(p-2)}{k_h}}\psi^{\frac{p(p-2)}{k_h}}(\tau)\,d\tau+\int_0^t \tau^{-\frac{N(q-1)}{k_h}}\psi^{\frac{p(q-1)}{k_h}}(\tau)\,d\tau\right\}\\
&\le \nu|||u_0|||^h_h+\nu\psi(t)\{t^h\psi^{p-2}(t)+t^{(\frac{1}{p})(k_h-N(q-1))}\psi^{q-1}(t)\}^{\frac{p}{k_h}}\\
&\le \nu|||u_0|||^h_h+\nu\,\delta^{\frac{p}{k_h}}\psi(t). \qquad (4.2.21)
\end{aligned}
$$

If $h=1$ take $\mathrm{sgn}_\eta u\xi^p$ as a testing function in (4.2.13). After a Steklov averaging process and standard calculations, we get

$$
\begin{aligned}
&\int_{B_2(x)}\int_0^{u(y,t)}\mathrm{sgn}_\eta u\,ds\xi^p\,dy+\int_0^t\int_{B_2(x)}|\nabla u|^p\mathrm{sgn}_\eta' u\xi^p\,dy\,d\tau\\
&\quad+p\int_0^t\int_{B_2(x)}|\nabla u|^{p-2}\nabla u\cdot\nabla\xi\xi^{p-1}\,dx\,d\tau\\
&=\int_{B_2(x)}\int_0^{u_0(y)}\mathrm{sgn}_\eta u\,ds\xi^p\,dy+\int_0^t\int_{B_2(x)}J^{\delta}_{0|t}(|u|^{q-1}u)\mathrm{sgn}_\eta u\xi^p\,dy\,d\tau.
\end{aligned}
$$

Discarding the second term on the left-hand side, which is nonnegative, letting $\eta\to 0$, then $\mathrm{sgn}_\eta u\to 1$ and using Lemma 4.2.4, we get

$$
\int_{B_1(x)}|u(y,t)|\,dy
$$

$$
\begin{aligned}
&\leq \int_{B_2(x)} |u_0(y)|\,dy + \nu \int_0^t \int_{B_2(x)} |\nabla u|^{p-1}\xi^{p-1}\,dy\,d\tau + \int_0^t \int_{B_2(x)} J^{\delta}_{0|t}(|u|^{q-1}u)\xi^p\,dy\,d\tau \\
&\leq \int_{B_2(x)} |u_0(y)|\,dy + \nu t^{\frac{1}{N(p-2)+p}} \left(\sup_{0<\tau<t} \int_{B_2(x)} |u|\,dy\right)^{1+\frac{p-2}{k}} \\
&\quad +\nu \left(\sup_{0<\tau<t} \int_{B_2(x)} |u|\,dy\right) t + \int_0^t \int_{B_2(x)} J^{\delta}_{0|t}(|u|^{q-1}u)\xi^p\,dy\,d\tau \qquad (4.2.22)\\
&\leq \int_{B_2(x)} |u_0(y)|\,dy + \nu t^{\frac{1}{N(p-2)+p}} \left(\sup_{0<\tau<t} \int_{B_2(x)} |u|\,dy\right)^{1+\frac{p-2}{k}} \\
&\quad +\nu \left(\sup_{0<\tau<t} \int_{B_2(x)} |u|\,dy\right) t + c\,t^{1-\gamma}||u(\cdot,t)||^{q-1}_{\infty,B_{2\rho}(x)} \int_0^t \int_{B_2(x)} |u(y,\tau)|\,dy\,d\tau,
\end{aligned}
$$

where $k = N(p-2)+p$.
We use (4.2.17) and (4.2.22) to obtain, that for all $t \in (0, t^*)$,

$$
\begin{aligned}
&\int_{B_1(x)} |u(y,t)|\,dy \\
&\leq \int_{B_2(x)} |u_0(y)|\,dy + \nu \left\{ \sup_{0<\tau<t} \sup_{x\in\mathbb{R}^N} \int_{B_2(x)} |u(y,\tau)|\,dy \right\} \\
&\quad \times \left\{ t^{\frac{1}{k}} \psi^{\frac{p-2}{k}} + \int_0^t \tau^{-\frac{N(q-1)}{k}}(\tau)\,d\tau + t \right\} \\
&\leq \nu |||u_0|||_1^1 + \nu\, \delta^{\frac{p}{k}} \psi(t) + \nu t \psi(t). \qquad (4.2.23)
\end{aligned}
$$

Let

$$\nu t^* \leq \frac{1}{2}.$$

By (4.2.21) and (4.2.23), we can determine $\delta = \delta(p,q,N)$ a priory depending only on the indicated quantities so that

$$\psi(t) \leq \nu |||u_0|||_h^h, \qquad \text{for any } 0 < t < t^*. \qquad (4.2.24)$$

The number t^* is still only qualitatively known. A quantitative lower bound can be found by substituting (4.2.24) into the definition of t^* in (4.2.18). It gives that (4.2.24) holds for all $0 < t < T_0$ where T_0 is the smallest root of

$$T_0 + T_0 |||u_0|||_h^{p-2} + T_0^{1-\frac{N(q-p+1)}{ph}} |||u_0|||_h^{q-1} = \nu^{-1},$$

for a constant $\gamma = \gamma(p,q,N) \geq 1$. substituting (4.2.24) into (4.2.17), we get (4.2.8).
Thus we have completed the proof of Theorem 4.2.2.

□

Proof of Lemma 4.2.3. Let $\rho > 0$, $\sigma \in (0, \frac{1}{2}]$ be fixed, let $k > 0$ to be chosen and for $n = 0, 1, 2, \cdots$ we set

$$\rho_n = \rho + \frac{\sigma}{2^n}\rho, \quad t_n = \frac{t}{2} - \left(\frac{\sigma}{2^{n+1}}\right)^p t, \quad k_n = k - \frac{k}{2^{n+1}},$$

$$B_n = B_{\rho_n}(x_0), \quad Q_n = B_n \times (t_n, t), \quad 0 < t_n < t \le T.$$

Let $\xi_n(x,t)$ be a smooth cutoff function in Q_n such that

$$\xi_n = 1 \qquad \text{on } Q_{n+1}, \quad 0 \le \frac{\partial \xi_n}{\partial t} \le \nu \frac{2^{np}}{\sigma^p t}, \quad |\nabla \xi_n| \le \nu \frac{2^{n+1}}{\sigma \rho}.$$

In Definition 4.2.2, we take the test function $\phi = (u - k_{n+1})_+^h \xi_n^p = [\max\{0, u - k_{n+1}\}]^h \xi_n^p$ to obtain

$$\begin{aligned}
&\int_{B_n(t')} u(u-k_{n+1})_+^h \xi_n^p \, dx + \int_{t_n}^{t'} \int_{B_n} -u \left[(u-k_{n+1})_+^h \xi_n^p\right]_t \, dx \, d\tau \\
&\quad + \int_{t_n}^{t'} \int_{B_n} |\nabla u|^{p-2} \nabla u \nabla \left[(u-k_{n+1})_+^h \xi_n^p\right] \, dx \, d\tau \\
&= \Gamma(\delta) \int_{t_n}^{t'} \int_{B_n} J_{0|t}^{\delta}(|u|^{q-1}u)(u-k_{n+1})_+^h \xi_n^p \, dx \, d\tau + \int_{B_n(t_n)} u(x,t_0)(u-k_{n+1})_+^h \xi_n^p \, dx.
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{1}{h+1} \int_{B_n(t')} (u-k_{n+1})_+^{h+1} \xi_n^p \, dx \\
&\quad + h \int_{t_n}^{t'} \int_{B_n} |\nabla u|^{p-2} \nabla u \nabla \left[(u-k_{n+1})_+\right] (u-k_{n+1})^{h-1} \xi_n^p \, dx \, d\tau \\
&\quad + p \int_{t_n}^{t'} \int_{B_n} (u-k_{n+1})_+^h \xi_n^{p-1} |\nabla u|^{p-2} \nabla u \nabla \xi_n \, dx \, d\tau \\
&= p \int_{t_n}^{t'} \int_{B_n} (u-k_{n+1})_+^{h+1} \xi_n^{p-1} \xi_{nt} \, dx \, d\tau + \Gamma(\delta) \int_{t_n}^{t'} \int_{B_n} J_{0|t}^{\delta}(|u|^{q-1}u)(u-k_{n+1})_+^h \xi_n^p \, dx \, d\tau,
\end{aligned} \tag{4.2.25}$$

where $t_n < t' < t$. By the Hölder inequality

$$\begin{aligned}
&\left| p \int_{t_n}^{t'} \int_{B_n} (u-k_{n+1})_+^h \xi_n^{p-1} |\nabla u|^{p-2} \nabla u \nabla \xi_n \, dx \, d\tau \right| \\
&\le p \int_{t_n}^{t'} \int_{B_n} (u-k_{n+1})_+^{\frac{(h-1)(p-1)}{p}} (u-k_{n+1})_+^{\frac{h+p-1}{p}} \xi_n^{p-1} |\nabla (u-k_{n+1})|^{p-1} |\nabla \xi_n| \, dx \, d\tau \\
&\le \frac{h}{2} \int_{t_n}^{t'} \int_{B_n} (u-k_{n+1})_+^{(h-1)} \xi_n^p |\nabla (u-k_{n+1})|^p \, dx \, d\tau \\
&\quad + \nu \int_{t_n}^{t'} \int_{B_n} (u-k_{n+1})_+^{(h+p-1)} |\nabla \xi_n|^p \, dx \, d\tau.
\end{aligned} \tag{4.2.26}$$

Notice that if $u > 2k_n$,

$$\frac{u}{2} > k_n \Rightarrow u - \frac{u}{2} > k_n \Rightarrow u - k_n > \frac{u}{2} \Rightarrow u < 2(u - k_n);$$

if $k_{n+1} \le u \le 2k_n$,

$$k_{n+1} - k_n = \frac{k}{2^{n+2}} > \frac{k_n}{2^{n+2}} > \frac{u}{2^{n+3}}$$

but

$$\frac{u}{2^{n+3}} < k_{n+1} - k_n < u - k_n,$$

we get

$$u < 2^{n+3}(u - k_n).$$

Thus by the Hölder inequality

$$\begin{aligned}
&\Gamma(\delta) \int_{t_n}^{t'} \int_{B_n} J_{0|t}^{\delta}(|u|^{q-1}u)(u-k_{n+1})_+^h \xi_n^p \, dx\, d\tau \qquad (4.2.27)\\
&\le \Gamma(\delta) \int_{t_n}^{t'} \int_{B_n} \left(\int_0^s (s-\sigma)^{-\gamma} u^q(\sigma)\, d\sigma \right) (u - K_{n+1})_+^h \, dx\, d\tau\\
&\le \Gamma(\delta) \int_{t_n}^{t'} || \int_0^s (s-\sigma)^{-\gamma} u^q(\sigma)\, d\sigma ||_{L_x^{h+1}(B_n)}^{h+1} \cdot ||(u-k_{n+1})_+^h||_{L_x^{\frac{h+1}{h}}(B_n)}^{\frac{h+1}{h}} \, dx\, d\tau\\
&\le \Gamma(\delta) \left(\int_{t_n}^{t'} || \int_0^s (s-\sigma)^{-\gamma} u^q(\sigma)\, d\sigma ||_{L_x^{h+1}(B_n)}^{h+1} \, dx\, d\tau \right)^{\frac{1}{h+1}} \times \left(\int_{t_n}^{t'} ||(u+k_{n+1})_+^h||_{L_x^{\frac{h+1}{h}}(B_n)}^{\frac{h+1}{h}} \, dx\, d\tau \right)^{\frac{h}{h+1}}\\
&\le \Gamma(\delta)\, c_1\, t^{1-\gamma} ||u(\cdot,t)||_{\infty, B_{2\rho}(x_0)}^{q-1} \left(\int_{t_n}^{t'} \int_{B_n} u^{h+1}\, dx\, d\tau \right)^{\frac{1}{h+1}} \times \left(\int_{t_n}^{t'} \int_{B_n} (u-k_{n+1})_+^{h+1}\, dx\, d\tau \right)^{\frac{h}{h+1}}\\
&\le \Gamma(\delta)\, c\, t^{1-\gamma} ||u(\cdot,t)||_{\infty, B_{2\rho}(x_0)}^{q-1} \left(\int_{t_n}^{t'} \int_{B_n} (u-k_n)_+^{h+1}\, dx\, d\tau \right)^{\frac{1}{h+1}} \times \left(\int_{t_n}^{t'} \int_{B_n} (u-k_n)_+^{h+1}\, dx\, d\tau \right)^{\frac{h}{h+1}}\\
&= \Gamma(\delta)\, c\, t^{1-\gamma} ||u(\cdot,t)||_{\infty, B_{2\rho}(x_0)}^{q-1} \int_{t_n}^{t'} \int_{B_n} (u-k_n)_+^{h+1}\, dx\, d\tau.
\end{aligned}$$

Substituting (4.2.26) and (4.2.27) into (4.2.25), we obtain

$$\begin{aligned}
&\frac{1}{h+1} \int_{B_n(t')} (u-k_{n+1})_+^{h+1} \xi_n^p \, dx\\
&\quad + h \int_{t_n}^{t'} \int_{B_n} |\nabla u|^{p-2} \nabla u \nabla \left[(u-k_{n+1})_+ \right] (u-k_{n+1})^{h-1} \xi_n^p \, dx\, d\tau\\
&\quad + \frac{h}{2} \int_{t_n}^{t'} \int_{B_n} |\nabla u|^{p-2} \nabla u \nabla \left[(u-k_{n+1})_+ \right] (u-k_{n+1})_+^{h-1} \xi_n^p \, dx\, d\tau\\
&\quad + \nu \int_{t_n}^{t'} \int_{B_n} (u-k_{n+1})_+^{h+p-1} |\nabla \xi_n|^p \, dx\, d\tau\\
&\le p \int_{t_n}^{t'} \int_{B_n} (u-k_{n+1})_+^{h+1} \xi_n^{p-1} \xi_{nt} \, dx\, d\tau + \Gamma(\delta)\, c\, t^{1-\gamma} ||u(\cdot,t)||_{\infty, B_{2\rho}(x_0)}^{q-1} \int_{t_n}^{t'} \int_{B_n} (u-k_n)_+^{h+1}\, dx\, d\tau.
\end{aligned}$$

Therefore

$$
\begin{aligned}
&\frac{1}{h+1}\int_{B_n(t')}(u-k_{n+1})_+^{h+1}\xi_n^p\,dx\\
&\quad+\frac{3h}{2}\int_{t_n}^{t'}\int_{B_n}|\nabla u|^{p-2}\nabla u\nabla\left[(u-k_{n+1})_+\right](u-k_{n+1})^{h-1}\xi_n^p\,dx\,d\tau\\
&\quad+\nu\int_{t_n}^{t'}\int_{B_n}(u-k_{n+1})_+^{h+p-1}|\nabla\xi_n|^p\,dx\,d\tau\\
&\le\left(\frac{p\nu 2^{np}}{\sigma^p t}+c\,t^{1-\gamma}\sup_{0<\tau<t}||u(\cdot,t)||^{q-1}_{\infty,B_{2\rho}(x_0)}\right)\int\int_{Q_n}(u-k_n)_+^{h+1}\,dx\,d\tau.
\end{aligned}
$$

Thus, by the following estimation

$$
\begin{aligned}
&\int\int_{Q_n}\left|\nabla\left[(u-k_{n+1})_+^{\frac{p-1+h}{p}}\xi_n\right]\right|^p dx\,d\tau\\
&=\int\int_{Q_n}\left|\frac{p-1+h}{p}(u-k_{n+1})_+^{\frac{h-1}{p}}\nabla\left[(u-k_{n+1})_+\right]\xi_n+(u-k_{n+1})_+^{\frac{p-1+h}{p}}\nabla\xi_n\right|^p dx\,d\tau\\
&\le\int\int_{Q_n}(u-k_{n+1})_+^{h-1}|\nabla(u-k_{n+1})_+|^p\xi_n^p\,dx\,d\tau+\int\int_{Q_n}(u-k_{n+1})_+^{p-1+h}|\nabla\xi_n|^p\,dx\,d\tau\\
&\le\int\int_{Q_n}(u-k_{n+1})_+^{h-1}|\nabla(u-k_{n+1})_+|^p\xi_n^p\,dx\,d\tau\\
&\quad+\left(\nu\frac{2^{n+1}}{\sigma\rho}\right)^p\sup_{0<\tau<t}||u(\cdot,\tau)||^{p-2}_{\infty,B_2(x_0)}\int\int_{Q_n}(u-k_{n+1})_+^{h+1}\,dx\,d\tau,
\end{aligned}
$$

we get

$$
\begin{aligned}
&\operatorname{ess}\sup_{t_n<\tau<t}\int_{B_n(\tau)}(u-k_{n+1})_+^{h+1}\xi_n^p\,dx+\int\int_{Q_n}\left|\nabla\left[(u-k_{n+1})_+^{\frac{h+p-1}{p}}\xi_n\right]\right|^p dx\,d\tau\\
&\le\frac{\nu 2^{np}}{\sigma^p t}(1+M)\int\int_{Q_n}(u-k_n)_+^{h+1}\,dx\,d\tau,
\end{aligned}
\tag{4.2.28}
$$

where

$$
M=\sup_{0<\tau<t}\tau\left\{||u(\cdot,\tau)||^{p-2}_{\infty,B_2(x_0)}\rho^{-p}+c\,\tau^{1-\gamma}\sup_{x\in B_2(x_0)}|u|^{q-1}\right\}.
$$

By the Gagliardo Nirenberg inequality [17, p.3], we choose
$v=(u-k_{n+1})_+^{\frac{h+p-1}{p}}\xi_n,\quad \alpha=\frac{p}{q},\quad q=\frac{pb}{h+p-1}$ and $s=\frac{(h+1)p}{h+p-1}$, we get

$$
\begin{aligned}
&\int\int_{Q_n}\xi_n^d(u-k_{n+1})_+^b\,dx\,d\tau\\
&\le\nu\int\int_{Q_n}\left|\nabla\left((u-k_{n+1})_+^{\frac{h+p-1}{p}}\xi_n\right)\right|^p dx\,d\tau\times\left(\operatorname{ess}\sup_{t_n<\tau<t}\int_{B_n(\tau)}(u-k_{n+1})_+^{h+1}\xi_n^p\,dx\right)^{\frac{p}{N}},
\end{aligned}
\tag{4.2.29}
$$

where $b = p + h - 1 + \frac{p(h+1)}{N}$ and d is large enough.
Set $A_n = \{(x,t) \in Q_{n-1} : u(x,t) \geq k_n\}$, $n = 1, 2, \cdots$ and observe that

$$\begin{aligned}
&\int\int_{Q_n} (u-k_n)_+^{h+1}\,dx\,d\tau \\
&\geq \int\int_{Q_n+1} (u-k_n)_+^{h+1}\chi_{[u>k_{n+1}]}\,dx\,d\tau \\
&\geq (k_{n+1}-k_n)^{h+1}|A_{n+1}| = \left(\frac{k}{2^{n+1}}\right)^{h+1}|A_{n+1}| \geq \nu 2^{-(h+1)n}|A_{n+1}|k^{h+1}. \qquad (4.2.30)
\end{aligned}$$

Combining (4.2.29), (4.2.30) with (4.2.28), we obtain

$$\begin{aligned}
&\int\int_{Q_{n+1}} (u-k_{n+1})_+^{h+1}\,dx\,d\tau \\
&\leq \int\int_{Q_n} (u-k_{n+1})_+^{h+1}\xi_n^{\frac{(h+1)d}{b}}\,dx\,d\tau \\
&\leq |A_{n+1}|^s \times \left(\int\int_{Q_n} \xi_n^d (u-k_{n+1})_+^b\,dx\,d\tau\right)^{\frac{h+1}{b}} \\
&\leq \nu|A_{n+1}|^s \times \left(\frac{2^{np}}{\sigma^p t}(1+M)\int\int_{Q_n}(u-k_n)_+^{h+1}\,dx\,d\tau\right)^{\left(\frac{N+p}{N}\right)\left(\frac{h+1}{b}\right)} \qquad (4.2.31)\\
&\leq \nu k^{-(b-h-1)\frac{(h+1)}{b}}C_0(\sigma^p t)^{-(1+\frac{p}{N})(h+1)b} \times (1+M)^{\left(1+\frac{p}{N}\right)\left(\frac{h+1}{b}\right)} \\
&\quad \times \left(\int\int_{Q_n}(u-k_n)_+^{h+1}\,dx\,d\tau\right)^{1+\frac{p(h+1)}{bN}},
\end{aligned}$$

where

$$s = \frac{N(p-2)+(h+1)p}{N(p+h-1)+p(h+1)} \text{ and } C_0 = 2^{\left(b-h-1+p+\frac{p^2}{N}\right)\left(\frac{h+1}{b}\right)}.$$

If k is chosen to satisfy

$$\int\int_{Q_0}\left(u-\frac{k}{2}\right)_+^{h+1}\,dx\,d\tau \leq \nu k^{\frac{N(b-h-1)}{p}}\left(\frac{1+M}{\sigma^p t}\right)^{-(1+\frac{N}{p})},$$

then by [42, Lemma 5.6, p.95], we get

$$\int\int_{Q_n}(u-k_n)_+^{h+1}\,dx\,d\tau \to 0 \quad \text{as } n \to \infty,$$

i.e., $||u_+||_{\infty,Q_\infty} \leq k$. Take

$$k = \nu\left\{\left(\frac{1+M}{\sigma^p t}\right)^{1+\frac{N}{p}}\int\int_{Q_0} u_+^{h+1}\,dx\,d\tau\right\}^{\frac{p}{N(b-h-1)}}.$$

It follows from this and the Hölder inequality that

$$\begin{aligned} ||u_+||_{\infty,Q_\infty} &\le \nu\left(\frac{1+M}{\sigma^p t}\right)^{\frac{(N+p)}{N(b-h-1)}}\left(||u_+||_{\infty,Q_0}\int\int_{Q_0} u_+^h\,dx\,d\tau\right)^{\frac{p}{N(b-h-1)}} \\ &\le \frac{1}{2}||u_+||_{\infty,Q_0}+\nu\left(\frac{1+M}{\sigma^p t}\right)^{\frac{(N+p)}{k_h}}\left(\int\int_{Q_0} u_+^h\,dx\,d\tau\right)^{\frac{p}{k_h}}. \end{aligned}$$

Similar to [1, p.393] we obtain

$$||u_+||_{\infty,B_1(x_0)\times(\frac{t}{2},t)} \le \nu\left(\frac{1+M}{\sigma^p t}\right)^{\frac{(N+p)}{k_h}}\left(\int\int_{Q_0} u_+^h\,dx\,d\tau\right)^{\frac{p}{k_h}}. \tag{4.2.32}$$

Also, taking $\phi=(-u-k_{n+1})_+^h\xi_n^p$, we get (4.2.32) for u_-. This implies (4.2.10).

Proof of Lemma 4.2.4. Let $\xi(x)$ be a piecewise smooth cutoff function in $B_{\frac{3}{2}}(x_0)$ such that $\xi=1$ on $B_1(x_0)$ and $|\nabla\xi|\le 2$. The calculations to follow are formal in which $u_+(-u_-)$ is required to be strictly positive. The calculations can be made rigorous by replacing $u_+(-u_-)$ with $(u_++\epsilon)(-u_-+\epsilon)$ and letting $\epsilon\to 0$. By the Hölder inequality, we have

$$\begin{aligned} &\int_0^t\int_{B_1(x_0)} |\nabla u|^{p-1}\xi^{p-1}\,dx\,d\tau \\ &= \int_0^t\int_{B_1(x_0)} \tau^{\beta}u_+^{-\frac{1}{p}}|\nabla u|^{p-1}\tau^{-\beta}u_+^{\frac{1}{p}}\,dx\,d\tau \\ &\le \left(\int_0^t\int_{B_1(x_0)} \tau^{\frac{\beta p}{p-1}}\frac{|\nabla u|^p}{u_+^{\frac{1}{p-1}}}\,dx\,d\tau\right)^{\frac{p-1}{p}}\times\left(\int_0^t\int_{B_1(x_0)}\tau^{-\beta p}u_+\,dx\,d\tau\right)^{\frac{1}{p}}. \end{aligned} \tag{4.2.33}$$

By (4.2.10), we can take the test function as

$$\phi = t^{\frac{\beta p}{p-1}}u_+^{1-\frac{1}{p-1}}\xi^p(x)$$

in (4.2.2) to obtain

$$\begin{aligned} &\int_{B_1(x_0)} ut^{\frac{\beta p}{p-1}}u_+^{1-\frac{1}{p-1}}\xi^p(x)\,dx-\int_0^t\int_{B_1(x_0)} u\left[\tau^{\frac{\beta p}{p-1}}u_+^{1-\frac{1}{p-1}}\xi^p(x)\right]_\tau\,dx\,d\tau \\ &+\int_0^t\int_{B_1(x_0)}|\nabla u|^{p-2}\nabla u\nabla\left[\tau^{\frac{\beta p}{p-1}}u_+^{1-\frac{1}{p-1}}\xi^p(x)\right]\,dx\,d\tau \\ &=\Gamma(\delta)\int_0^t\int_{B_1(x_0)} J_{0|t}^{\delta}(|u|^{q-1}u)\tau^{\frac{\beta p}{p-1}}u_+^{1-\frac{1}{p-1}}\xi^p(x)\,dx\,d\tau+\int_{B_1(x_0)}u(x,t_0)t_0^{\frac{\beta p}{p-1}}u_+^{1-\frac{1}{p-1}}\xi^p(x)\,dx. \end{aligned} \tag{4.2.34}$$

We have

$$\int_0^t\int_{B_1(x_0)}|\nabla u|^{p-2}\nabla u\nabla\left[\tau^{\frac{\beta p}{p-1}}u_+^{1-\frac{1}{p-1}}\xi^p(x)\right]\,dx\,d\tau$$

$$\leq \int_0^t \int_{B_1(x_0)} \tau^{\frac{\beta p}{p-1}} \frac{|\nabla u|^p}{u_+^{\frac{1}{p-1}}} \, dx \, d\tau + p \int_0^t \int_{B_1(x_0)} \tau^{\frac{\beta p}{p-1}} |\nabla u|^{p-1} u_+^{1-\frac{1}{p-1}} \xi^{p-1} |\nabla \xi| \, dx \, d\tau$$
$$\leq \int_0^t \int_{B_1(x_0)} \tau^{\frac{\beta p}{p-1}} \frac{|\nabla u|^p}{u_+^{\frac{1}{p-1}}} \, dx \, d\tau + p \int_0^t \int_{B_1(x_0)} \frac{|\nabla u|^{p-1}}{u^{\frac{1}{p}}} \tau^{\beta} u_+^{\frac{1}{p}} u_+^{1-\frac{1}{p-1}} \tau^{\frac{\beta}{p-1}} \, dx \, d\tau$$
$$\leq \int_0^t \int_{B_1(x_0)} \tau^{\frac{\beta p}{p-1}} \frac{|\nabla u|^p}{u_+^{\frac{1}{p-1}}} \, dx \, d\tau + p \int_0^t \int_{B_1(x_0)} \frac{|\nabla u|^p}{u_+^{\frac{1}{p-1}}} \tau^{\frac{\beta p}{p-1}} \, dx \, d\tau$$
$$+ p \int_0^t \int_{B_1(x_0)} u_+^{p-\frac{1}{p-1}} \tau^{\frac{\beta p}{p-1}} \, dx \, d\tau$$
$$= C \int_0^t \int_{B_1(x_0)} \tau^{\frac{\beta p}{p-1}} \frac{|\nabla u|^p}{u_+^{\frac{1}{p-1}}} \, dx \, d\tau + \nu \int_0^t \int_{B_1(x_0)} u_+^{p-\frac{1}{p-1}} \tau^{\frac{\beta p}{p-1}} \, dx \, d\tau.$$

By the Hölder inequality

$$\Gamma(\delta) \int_0^t \int_{B_1(x_0)} J_{0|t}^{\delta}(|u|^{q-1}u) \tau^{\frac{\beta p}{p-1}} u_+^{1-\frac{1}{p-1}} \xi^p(x) \, dx \, d\tau$$
$$\leq \Gamma(\delta) \int_0^t ||\tau^{\frac{\beta p}{2p-3}} \int_0^s (s-\sigma)^{-\gamma} u^q(\sigma) \, d\sigma||_{L_x^{\frac{2p-3}{p-1}}(B_1(x_0))}^{\frac{2p-3}{p-1}} \times ||\tau^{\frac{\beta p(p-2)}{(2p-3)(p-1)}} u_+^{1-\frac{1}{p-1}}||_{L_x^{\frac{2p-3}{p-2}}(B_1(x_0))}^{\frac{2p-3}{p-2}} \, dx \, d\tau$$
$$\leq \Gamma(\delta) \left(\int_0^t ||\tau^{\frac{\beta p}{2p-3}} \int_0^s (s-\sigma)^{-\gamma} u^q(\sigma) \, d\sigma||_{L_x^{\frac{2p-3}{p-1}}(B_1(x_0))}^{\frac{2p-3}{p-1}} \, dx \, d\tau \right)^{\frac{p-1}{2p-3}}$$
$$\times \left(\int_0^t ||\tau^{\frac{\beta p(p-2)}{(2p-3)(p-1)}} u_+^{1-\frac{1}{p-1}}||_{L_x^{\frac{2p-3}{p-2}}(B_1(x_0))}^{\frac{2p-3}{p-2}} \, dx \, d\tau \right)^{\frac{p-2}{2p-3}}$$
$$\leq \Gamma(\delta)\, c_1\, t^{1-\gamma} ||u(\cdot,t)||_{\infty, B_{2\rho}(x_0)}^{q-1} \left(\int_0^t \int_{B_{\frac{3}{2}}(x_0)} \tau^{\frac{\beta p}{p-1}} u_+^{\frac{2p-3}{p-1}} \, dx \, d\tau \right)^{\frac{p-1}{2p-3}}$$
$$\times \left(\int_0^t \int_{B_{\frac{3}{2}}(x_0)} \tau^{\frac{\beta p}{p-1}} u_+^{\frac{2p-3}{p-1}} \, dx \, d\tau \right)^{\frac{p-2}{2p-3}}$$
$$\leq \Gamma(\delta)\, c\, t^{1-\gamma} ||u(\cdot,t)||_{\infty, B_{2\rho}(x_0)}^{q-1} \int_0^t \int_{B_{\frac{3}{2}}(x_0)} \tau^{\frac{\beta p}{p-1}} u_+^{2-\frac{1}{p-1}} \, dx \, d\tau.$$

Moreover

$$\int_0^t \int_{B_1(x_0)} u \left[\tau^{\frac{\beta p}{p-1}} u_+^{1-\frac{1}{p-1}} \xi^p(x) \right]_{\tau} \, dx \, d\tau$$
$$\leq \nu \int_0^t \int_{B_1(x_0)} u \tau^{\frac{\beta p}{p-1}-1} u_+^{1-\frac{1}{p-1}} \, dx \, d\tau + \nu \int_0^t \int_{B_1(x_0)} u \tau^{\frac{\beta p}{p-1}} u_+^{-\frac{1}{p-1}} u_t \, dx \, d\tau$$
$$\leq \nu \int_0^t \int_{B_1(x_0)} u \tau^{\frac{\beta p}{p-1}-1} u_+^{1-\frac{1}{p-1}} \, dx \, d\tau + \nu \int_0^t \int_{B_1(x_0)} \tau^{\frac{\beta p}{p-1}} u_+^{1-\frac{1}{p-1}} \operatorname{div}\left(|\nabla u|^{p-2} \nabla u \right) \, dx \, d\tau$$

$$
\begin{aligned}
&+\Gamma(\delta)\int_0^t\int_{B_1(x_0)} J^{\delta}_{0|t}(|u|^{q-1}u)\tau^{\frac{\beta p}{p-1}}u_+^{1-\frac{1}{p-1}}\xi^p(x)\,dx\,d\tau\\
\le\ &\nu\int_0^t\int_{B_1(x_0)} u\tau^{\frac{\beta p}{p-1}-1}u_+^{1-\frac{1}{p-1}}\,dx\,d\tau+\nu\int_0^t\int_{B_1(x_0)}\tau^{\frac{\beta p}{p-1}}u_+^{-\frac{1}{p-1}}|\nabla u|^p\,dx\,d\tau\\
&+\Gamma(\delta)\int_0^t\int_{B_1(x_0)} J^{\delta}_{0|t}(|u|^{q-1}u)\tau^{\frac{\beta p}{p-1}}u_+^{1-\frac{1}{p-1}}\xi^p(x)\,dx\,d\tau\\
&+\nu\int_0^t\int_{B_1(x_0)} u\tau^{\frac{\beta p}{p-1}-1}u_+^{1-\frac{1}{p-1}}\,dx\,d\tau+\nu\int_0^t\int_{B_1(x_0)}\tau^{\frac{\beta p}{p-1}}\frac{|\nabla u|^p}{u_+^{\frac{1}{p-1}}}\,dx\,d\tau\\
&+\Gamma(\delta)\int_0^t\int_{B_1(x_0)} J^{\delta}_{0|t}(|u|^{q-1}u)\tau^{\frac{\beta p}{p-1}}u_+^{1-\frac{1}{p-1}}\xi^p(x)\,dx\,d\tau,
\end{aligned}
$$

where we have used the fact that $u_t=\operatorname{div}(|\nabla u|^{p-2}\nabla u)+\Gamma(\delta)J^{\delta}_{0|t}(|u|^{q-1}u)$. Actually, the above estimate holds for any weak solution u by the density argument.
Hence (4.2.34) gives

$$
\begin{aligned}
&\int_0^t\int_{B_1(x_0)}\tau^{\frac{\beta p}{p-1}}\frac{|\nabla u|^p}{u_+^{\frac{1}{p-1}}}\,dx\,d\tau\\
\le\ &\nu\int_0^t\int_{B_{\frac{3}{2}}(x_0)}\tau^{\frac{\beta p}{p-1}}u_+^{p-\frac{1}{p-1}}\,dx\,d\tau+\ \nu\int_0^t\int_{B_{\frac{3}{2}}(x_0)}\tau^{\frac{\beta p}{p-1}-1}u_+^{2-\frac{1}{p-1}}\,dx\,d\tau\\
&+c\,t^{1-\gamma}||u(\cdot,t)||^{q-1}_{\infty,B_{2\rho}(x_0)}\int_0^t\int_{B_{\frac{3}{2}}(x_0)}\tau^{\frac{\beta p}{p-1}}u_+^{2-\frac{1}{p-1}}\,dx\,d\tau,
\end{aligned}
$$

therefore

$$
\begin{aligned}
&\int_0^t\int_{B_1(x_0)}\tau^{\frac{\beta p}{p-1}}\frac{|\nabla u|^p}{u_+^{\frac{1}{p-1}}}\,dx\,d\tau\\
\le\ &\nu(1+M)\int_0^t\int_{B_{\frac{3}{2}}(x_0)}\tau^{\frac{\beta p}{p-1}-1}u_+^{2-\frac{1}{p-1}}\,dx\,d\tau\\
\le\ &\nu(1+M)G(t)\int_0^t\tau^{\frac{\beta p}{p-1}-1}||u_+(\cdot,\tau)||^{1-\frac{1}{p-1}}_{\infty,B_{\frac{3}{2}}(x_0)}\,d\tau\,d\tau,
\end{aligned}
\tag{4.2.35}
$$

where

$$
M=\sup_{0<\tau<t}\tau\left\{||u(\cdot,\tau)||^{p-2}_{\infty,B_2(x_0)}\rho^{-p}+\tau^{1-\gamma}\sup_{x\in B_2(x_0)}|u|^{q-1}\right\},
$$

and

$$
G(t)=\sup_{0<\tau<t}\left(\int_{B_2(x_0)}|u(x,\tau)|^h\,dx\right)^{\frac{1}{h}}.
$$

Notice that by (4.2.10),

$$
||u(\cdot,t)||_{\infty,B_\rho(x_0)}\le\nu t^{-\frac{N+p}{k_h}}\left(\int_0^t\int_{B_{2\rho}(x_0)}|u|^h\,dx\,d\tau\right)^{\frac{p}{k_h}}\le\nu t^{-\frac{N}{k_h}}G(t)^{\frac{ph}{k_h}},
$$

and

$$\int_0^t \int_{B_1(x_0)} \tau^{-\beta p} u_+ \, dx \, d\tau \leq \int_0^t \int_{B_2(x_0)} \tau^{-\beta p} u_+ \, dx \, d\tau \leq \nu t^{1-\beta p} G(t). \tag{4.2.36}$$

Substituting (4.2.35) and (4.2.36) into (4.2.33), we get

$$\begin{aligned}
&\int_0^t \int_{B_1(x_0)} |\nabla u|^{p-1} \xi^{p-1} \, dx \, d\tau \\
&\leq \left(\nu(1+M)G(t) \int_0^t \tau^{\frac{\beta p}{p-1}-1} \left(\nu t^{-\frac{N}{k_h}} G(t)^{\frac{ph}{k_h}} \right)^{1-\frac{1}{p-1}} d\tau \right)^{\frac{p-1}{p}} \times \left(\nu t^{1-\beta p} G(t) \right)^{\frac{1}{p}} \\
&\leq \nu G(t)^{1+\frac{h(p-2)}{k_h}} \left(t^{\frac{\beta p}{p-1} - \frac{N(p-2)}{k_h(p-1)}} \right)^{\frac{p-1}{p}} \times t^{\frac{1}{p}-\beta} \\
&= \nu G(t)^{1+\frac{h(p-2)}{k_h}} t^{\frac{k_h - N(p-2)}{k_h p}} = \nu G(t)^{1+\frac{h(p-2)}{k_h}} t^{\frac{h}{k_h}}.
\end{aligned}$$

Remark : If $h = 1$, for any constant $\rho \geq 1$ we can prove

$$\int_0^t \int_{B_\rho(x_0)} |\nabla u|^{p-1} \, dx \, d\tau \leq \nu t^{\frac{1}{k}} G(t)^{1+\frac{(p-2)}{k}},$$

where $k = N(p-2) + p$.

4.3 Global nonexistence

In this section we present the proof of Theorem 4.1.1.

Proof of Theorem 4.1.1. Let u be a nonnegative global weak solution of (4.1.1), we have

$$\Gamma(\delta) \int_0^\infty \int_\Omega J^\delta_{0|t}(u^q) \varphi \, dx \, dt = \int_0^\infty \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx \, dt - \int_0^\infty \int_\Omega u \varphi_t \, dx \, dt - \int_\Omega u_0(x) \varphi(x,0) \, dx.$$

where Ω is any bounded open set in $\mathbb{R}^n$. Here

$$\varphi(x,t) = D^\delta_{t|T} \left(\tilde{\varphi}(x,t) \right) := D^\delta_{t|T} \left(\varphi_1^\ell(x) \varphi_2(t) \right) \quad \text{with} \quad \varphi_1(x) := \Phi\left(|x|/B \right), \; \varphi_2(t) := \left(1 - \frac{t}{T} \right)_+^\eta, \tag{4.3.1}$$

where $D^\delta_{t|T}$ is given by (1.1.1), $T > 0$, $\ell, \eta \gg 1$ and $\Phi \in C^\infty(\mathbb{R}_+)$ is a cut-off non-increasing function such that

$$\Phi(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq 1, \\ 0 & \text{if } r \geq 2, \end{cases}$$

$0 \leq \Phi \leq 1$ and $|\Phi'(r)| \leq C_1/r$ for all $r > 0$. The constant $B > 0$ in the definition of φ_1 is fixed and will be chosen later. Then, we obtain

$$\Gamma(\delta) \int_0^T \int_{\Omega(B)} J^\delta_{0|t}(u^q) D^\delta_{t|T} \tilde{\varphi} \, dx \, dt + \int_{\Omega(B)} u_0(x) D^\delta_{t|T} \tilde{\varphi}(0,x) \, dx$$

$$= \int_0^T \int_{\Omega(B)} |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx\, dt - \int_0^T \int_{\Omega(B)} u\, \partial_t \varphi \, dx\, dt,$$

where $\Omega(B) := \{x \in \mathbb{R}^n;\ |x| < 2B\}$. From (1.1.3) and (1.1.7), we conclude that

$$\int_0^T \int_{\Omega(B)} D^\delta_{0|t} J^\delta_{0|t}(u^q)\tilde{\varphi}\, dx\, dt + C\, T^{-\delta} \int_{\Omega(B)} u_0(x)\varphi_1^\ell(x)\, dx$$
$$= C \int_0^T \int_{\Omega(B)} |\nabla u|^{p-2}\nabla u \nabla \varphi\, dx\, dt - C \int_0^T \int_{\Omega(B)} u\, \partial_t \varphi\, dx\, dt,$$

where $D^\delta_{0|t}$ is defined in (1.1.1). Moreover, using (1.1.5) and the nonnegativity of u_0 and u, it follows

$$\int_0^T \int_{\Omega(B)} u^q \tilde{\varphi}\, dx\, dt \le C \int_0^T \int_{\Omega(B)} |\nabla u|^{p-1} |\nabla \varphi|\, dx\, dt + C \int_0^T \int_{\Omega(B)} u|\partial_t \varphi|\, dx\, dt := I_1 + I_2. \quad (4.3.2)$$

Next we observe that by introducing the term $\tilde{\varphi}^{1/q}\tilde{\varphi}^{-1/q}$ in I_2 and applying Young's inequality

$$ab \le \frac{1}{8} a^q + C(q) b^{q'}, \quad a \ge 0,\ b \ge 0,\ q' = q/(q-1),$$

we obtain

$$I_2 \le \frac{1}{8} \int_0^T \int_{\Omega(B)} u^q \tilde{\varphi}\, dx\, dt + C \int_0^T \int_{\Omega(B)} \varphi_1^\ell \varphi_2^{-1/(q-1)} \left(D^{1+\delta}_{t|T} \varphi_2 \right)^{q'} dx\, dt. \quad (4.3.3)$$

In order to obtain a similar estimation on I_1, let $\alpha < 0$ be an auxiliary constant such that $\alpha > \max\{-1, 1-p\}$ and let

$$u_\epsilon(x,t) = u(x,t) + \epsilon, \quad \epsilon > 0.$$

As u is a weak solution, by taking $\varphi_\epsilon(x,t) = u_\epsilon^\alpha(x,t)\varphi(x,t)$ as a test function where φ is given in (4.3.1), we have

$$\begin{aligned}\Gamma(\delta) \int_0^\infty \int_{\Omega(B)} J^\delta_{0|t}(u^q) u_\epsilon^\alpha \varphi\, dx\, dt \;&=\; \int_0^\infty \int_{\Omega(B)} |\nabla u|^{p-2} \nabla u \nabla (u_\epsilon^\alpha \varphi)\, dx\, dt \\ &\quad - \int_0^\infty \int_{\Omega(B)} u \partial_t (u_\epsilon^\alpha \varphi)\, dx\, dt - \int_{\Omega(B)} u_0(x) u_\epsilon^\alpha(x,0)\varphi(x,0)\, dx.\end{aligned}$$

Then, using the fact that

$$\nabla(u_\epsilon^\alpha \varphi) = \alpha u_\epsilon^{\alpha-1} \nabla(u)\varphi + u_\epsilon^\alpha \nabla(\varphi) \quad \text{and} \quad \partial_t(u_\epsilon^\alpha \varphi) = \alpha u_\epsilon^{\alpha-1}\partial_t(u)\varphi + u_\epsilon^\alpha \partial_t(\varphi),$$

we get

$$\begin{aligned}\Gamma(\delta) \int_0^\infty \int_{\Omega(B)} J^\delta_{0|t}(u^q) u_\epsilon^\alpha \varphi\, dx\, dt \;&=\; \alpha \int_0^\infty \int_{\Omega(B)} |\nabla u|^p u_\epsilon^{\alpha-1} \varphi\, dx\, dt + \int_0^\infty \int_{\Omega(B)} (|\nabla u|^{p-2} \nabla u \nabla \varphi) u_\epsilon^\alpha\, dx\, dt \\ &\quad - \alpha \int_0^\infty \int_{\Omega(B)} u u_\epsilon^{\alpha-1} \partial_t(u)\varphi\, dx\, dt - \int_0^\infty \int_{\Omega(B)} u u_\epsilon^\alpha \partial_t \varphi\, dx\, dt\end{aligned}$$

$$-\int_{\Omega(B)} u_0(x)u_\epsilon^\alpha(x,0)\varphi(x,0)\,dx. \tag{4.3.4}$$

Using the definition of u_ϵ and integrating by parts, we obtain

$$\begin{aligned}
J_1 &:= \int_0^\infty\int_{\Omega(B)} uu_\epsilon^{\alpha-1}\partial_t(u)\varphi\,dx\,dt\\
&= \int_0^\infty\int_{\Omega(B)} (u_\epsilon-\epsilon)u_\epsilon^{\alpha-1}\partial_t(u)\varphi\,dx\,dt\\
&= \int_0^\infty\int_{\Omega(B)} u_\epsilon^{\alpha}\partial_t(u_\epsilon)\varphi\,dx\,dt-\epsilon\int_0^\infty\int_{\Omega(B)} u_\epsilon^{\alpha-1}\partial_t(u_\epsilon)\varphi\,dx\,dt\\
&= \frac{1}{\alpha+1}\int_0^\infty\int_{\Omega(B)} \partial_t(u_\epsilon^{\alpha+1})\varphi\,dx\,dt-\frac{\epsilon}{\alpha}\int_0^\infty\int_{\Omega(B)} \partial_t(u_\epsilon^{\alpha})\varphi\,dx\,dt\\
&= -\frac{1}{\alpha+1}\int_0^\infty\int_{\Omega(B)} u_\epsilon^{\alpha+1}\partial_t\varphi\,dx\,dt-\frac{1}{\alpha+1}\int_{\Omega(B)} u_\epsilon^{\alpha+1}(x,0)\varphi(x,0)\,dx\\
&\quad +\frac{\epsilon}{\alpha}\int_0^\infty\int_{\Omega(B)} u_\epsilon^{\alpha}\partial_t\varphi\,dx\,dt+\frac{\epsilon}{\alpha}\int_{\Omega(B)} u_\epsilon^{\alpha}(x,0)\varphi(x,0)\,dx.
\end{aligned}$$

Similarly,

$$\begin{aligned}
J_2 &= \int_0^\infty\int_{\Omega(B)} uu_\epsilon^{\alpha}\partial_t\varphi\,dx\,dt\\
&= \int_0^\infty\int_{\Omega(B)} (u_\epsilon-\epsilon)u_\epsilon^{\alpha}\partial_t\varphi\,dx\,dt\\
&= \int_0^\infty\int_{\Omega(B)} u_\epsilon^{\alpha+1}\partial_t\varphi\,dx\,dt-\epsilon\int_0^\infty\int_{\Omega(B)} u_\epsilon^{\alpha}\partial_t\varphi\,dx\,dt,
\end{aligned}$$

and

$$\begin{aligned}
J_3 &= \int_{\Omega(B)} u_0(x)u_\epsilon^\alpha(x,0)\varphi(x,0)\,dx\\
&= \int_{\Omega(B)} (u_\epsilon(x,0)-\epsilon)u_\epsilon^\alpha(x,0)\varphi(x,0)\,dx\\
&= \int_{\Omega(B)} u_\epsilon^{\alpha+1}(x,0)\varphi(x,0)\,dx-\epsilon\int_{\Omega(B)} u_\epsilon^{\alpha}(x,0)\varphi(x,0)\,dx.
\end{aligned}$$

Using J_1,J_2, and J_3, it follows from (4.3.4) that

$$\begin{aligned}
\Gamma(\delta)\int_0^\infty\int_{\Omega(B)} J_{0|t}^\delta(u^q)u_\epsilon^\alpha\varphi\,dx\,dt &= \alpha\int_0^\infty\int_{\Omega(B)} |\nabla u|^p u_\epsilon^{\alpha-1}\varphi\,dx\,dt\\
&\quad +\int_0^\infty\int_{\Omega(B)} (|\nabla u|^{p-2}\nabla u\nabla\varphi)u_\epsilon^\alpha\,dx\,dt\\
&\quad -\frac{1}{\alpha+1}\int_0^\infty\int_{\Omega(B)} u_\epsilon^{\alpha+1}\partial_t\varphi\,dx\,dt
\end{aligned}$$

$$-\frac{1}{\alpha+1}\int_{\Omega(B)} u_\epsilon^{\alpha+1}(x,0)\varphi(x,0)\,dx,$$

then

$$\begin{aligned}
&\Gamma(\delta)\int_0^\infty\int_{\Omega(B)} J^\delta_{0|t}(u^q)u_\epsilon^\alpha\varphi\,dx\,dt+|\alpha|\int_0^\infty\int_{\Omega(B)}|\nabla u|^p u_\epsilon^{\alpha-1}\varphi\,dx\,dt\\
&=\int_0^\infty\int_{\Omega(B)}(|\nabla u|^{p-2}\nabla u\nabla\varphi)u_\epsilon^\alpha\,dx\,dt-\frac{1}{\alpha+1}\int_0^\infty\int_{\Omega(B)}u_\epsilon^{\alpha+1}\partial_t\varphi\,dx\,dt\\
&\ -\frac{1}{\alpha+1}\int_{\Omega(B)}u_\epsilon^{\alpha+1}(x,0)\varphi(x,0)\,dx\\
&\le\int_0^\infty\int_{\Omega(B)}|\nabla u|^{p-1}|\nabla\varphi|u_\epsilon^\alpha\,dx\,dt+\frac{1}{\alpha+1}\int_0^\infty\int_{\Omega(B)}u_\epsilon^{\alpha+1}|\partial_t\varphi|\,dx\,dt\\
&=\int_0^\infty\int_{\Omega(B)}(|\nabla u|^{p-1}u_\epsilon^{\frac{(\alpha-1)(p-1)}{p}}\varphi^{\frac{p-1}{p}})(|\nabla\varphi|u_\epsilon^{\alpha-\frac{(\alpha-1)(p-1)}{p}}\varphi^{-\frac{p-1}{p}})\,dx\,dt\\
&\ +\frac{1}{\alpha+1}\int_0^\infty\int_{\Omega(B)}u_\epsilon^{\alpha+1}|\partial_t\varphi|\,dx\,dt\\
&\le\frac{|\alpha|}{2}\int_0^\infty\int_{\Omega(B)}|\nabla u|^p u_\epsilon^{\alpha-1}\varphi\,dx\,dt+C(\alpha)\int_0^\infty\int_{\Omega(B)}u_\epsilon^{p-1+\alpha}|\nabla\varphi|^p\varphi^{1-p}\,dx\,dt\\
&\ +\frac{1}{\alpha+1}\int_0^\infty\int_{\Omega(B)}u_\epsilon^{\alpha+1}|\partial_t\varphi|\,dx\,dt,
\end{aligned}$$

where we have used the following Young inequality

$$ab\le\frac{|\alpha|}{2}\,a^{\frac{p}{p-1}}+C(\alpha)\,b^p,\qquad a>0,\ b>0. \tag{4.3.5}$$

Therefore, using the positivity of the solution u, we conclude that

$$\int_0^\infty\int_{\Omega(B)}|\nabla u|^p u_\epsilon^{\alpha-1}\varphi\,dx\,dt\le C\int_0^\infty\int_{\Omega(B)}u_\epsilon^{p-1+\alpha}|\nabla\varphi|^p\varphi^{1-p}\,dx\,dt+C\int_0^\infty\int_{\Omega(B)}u_\epsilon^{\alpha+1}|\partial_t\varphi|\,dx\,dt.$$

Using the last inequality and Young's inequality (4.3.5), we get

$$\begin{aligned}
\int_0^\infty\int_{\Omega(B)}|\nabla u|^{p-1}|\nabla\varphi|\,dx\,dt &= \int_0^\infty\int_{\Omega(B)}\left(|\nabla u|^{p-1}u_\epsilon^{\frac{(\alpha-1)(p-1)}{p}}\varphi^{\frac{p-1}{p}}\right)\left(u_\epsilon^{\frac{(1-\alpha)(p-1)}{p}}\varphi^{\frac{1-p}{p}}|\nabla\varphi|\right)dx\,dt\\
&\le \int_0^\infty\int_{\Omega(B)}|\nabla u|^p u_\epsilon^{\alpha-1}\varphi\,dx\,dt\\
&\quad + C\int_0^\infty\int_{\Omega(B)}u_\epsilon^{(1-\alpha)(p-1)}\varphi^{1-p}|\nabla\varphi|^p\,dx\,dt\\
&\le C\int_0^\infty\int_{\Omega(B)}u_\epsilon^{p-1+\alpha}|\nabla\varphi|^p\varphi^{1-p}\,dx\,dt+C\int_0^\infty\int_{\Omega(B)}u_\epsilon^{\alpha+1}|\partial_t\varphi|\,dx\,dt\\
&\quad + C\int_0^\infty\int_{\Omega(B)}u_\epsilon^{(1-\alpha)(p-1)}\varphi^{1-p}|\nabla\varphi|^p\,dx\,dt.
\end{aligned}\tag{4.3.6}$$

Apply the Fatou and Lebesgue theorems, as $\epsilon \to 0$, we obtain

$$\begin{aligned} I_1 &\leq C\int_0^\infty \int_{\Omega(B)} u^{p-1+\alpha}|\nabla\varphi|^p\varphi^{1-p}\,dx\,dt \\ &+C\int_0^\infty \int_{\Omega(B)} u^{\alpha+1}|\partial_t\varphi|\,dx\,dt + C\int_0^\infty \int_{\Omega(B)} u^{(1-\alpha)(p-1)}\varphi^{1-p}|\nabla\varphi|^p\,dx\,dt =: K_1+K_2+K_3. \end{aligned}$$

At this stage, we use Young's inequality and the fact that $\nabla(\varphi_1^\ell) = \ell\varphi^{\ell-1}\nabla\varphi_1$ to estimate K_1, K_2 and K_3, precisely, we have :

$$\begin{aligned} K_1 &= \int_0^T \int_{\Omega(B)} (u^{p-1+\alpha}\tilde{\varphi}^{(p-1+\alpha)/q})(C\,\tilde{\varphi}^{-(p-1+\alpha)/q}|\nabla\varphi|^p\varphi^{1-p})\,dx\,dt \\ &\leq \frac{1}{8}\int_0^T \int_{\Omega(B)} u^q\tilde{\varphi} + C\int_0^T \int_{\Omega(B)} \varphi_1^{\frac{\ell(q-p+1-\alpha)-pq}{q-p+1-\alpha}}|\nabla\varphi_1|^{\frac{pq}{q-p+1-\alpha}}\varphi_2^{-\frac{p-1+\alpha}{q-p+1-\alpha}}\left(D_{t|T}^\delta\varphi_2\right)^{\frac{q}{q-p+1-\alpha}} \end{aligned}$$

$$\begin{aligned} K_2 &= \int_0^T \int_{\Omega(B)} (u^{\alpha+1}\tilde{\varphi}^{(\alpha+1)/q})(C\,\tilde{\varphi}^{-(\alpha+1)/q}|\partial_t\varphi|)\,dx\,dt \\ &\leq \frac{1}{8}\int_0^T \int_{\Omega(B)} u^q\tilde{\varphi}\,dx\,dt + C\int_0^T \int_{\Omega(B)} \varphi_1^\ell\varphi_2^{-\frac{\alpha+1}{q-1-\alpha}}\left(D_{t|T}^{1+\delta}\varphi_2\right)^{\frac{q}{q-1-\alpha}}\,dx\,dt, \end{aligned}$$

and

$$\begin{aligned} K_3 &= \int_0^T \int_{\Omega(B)} (u^{(1-\alpha)(p-1)}\tilde{\varphi}^{(1-\alpha)(p-1)/q})(C\,\tilde{\varphi}^{-(1-\alpha)(p-1)/q}\varphi^{1-p}|\nabla\varphi|^p)\,dx\,dt \\ &\leq \frac{1}{8}\int_0^T \int_{\Omega(B)} u^q\tilde{\varphi}\,dx\,dt \\ &\quad +C\int_0^T \int_{\Omega(B)} \varphi_1^{\frac{\ell(q-(1-\alpha)(p-1))-pq}{q-(1-\alpha)(p-1)}}|\nabla\varphi_1|^{\frac{pq}{q-p+1-\alpha}}\varphi_2^{-\frac{(1-\alpha)(p-1)}{q-(1-\alpha)(p-1)}}\left(D_{t|T}^\delta\varphi_2\right)^{\frac{q}{q-(1-\alpha)(p-1)}}\,dx\,dt. \end{aligned}$$

Therefore, we conclude that

$$\begin{aligned} I_1 &\leq \frac{3}{8}\int_0^T \int_{\Omega(B)} u^q\tilde{\varphi}\,dx\,dt \\ &+ C\int_0^T \int_{\Omega(B)} \varphi_1^\ell\varphi_2^{-\frac{\alpha+1}{q-1-\alpha}}\left(D_{t|T}^{1+\delta}\varphi_2\right)^{\frac{q}{q-1-\alpha}}\,dx\,dt \\ &+ C\int_0^T \int_{\Omega(B)} \varphi_1^{\frac{\ell(q-p+1-\alpha)-pq}{q-p+1-\alpha}}|\nabla\varphi_1|^{\frac{pq}{q-p+1-\alpha}}\varphi_2^{-\frac{p-1+\alpha}{q-p+1-\alpha}}\left(D_{t|T}^\delta\varphi_2\right)^{\frac{q}{q-p+1-\alpha}}\,dx\,dt \\ &+ C\int_0^T \int_{\Omega(B)} \varphi_1^{\frac{\ell(q-(1-\alpha)(p-1))-pq}{q-(1-\alpha)(p-1)}}|\nabla\varphi_1|^{\frac{pq}{q-p+1-\alpha}}\varphi_2^{-\frac{(1-\alpha)(p-1)}{q-(1-\alpha)(p-1)}}\left(D_{t|T}^\delta\varphi_2\right)^{\frac{q}{q-(1-\alpha)(p-1)}}\,dx\,dt. \end{aligned}$$

Using the estimations of I_1 and I_2 into (4.3.2), we get

$$\int_0^T \int_{\Omega(B)} u^q\tilde{\varphi}\,dx\,dt \tag{4.3.7}$$

$$
\begin{aligned}
&\leq C\int_0^T\int_{\Omega(B)}\varphi_1^{\ell}\varphi_2^{-1/(q-1)}\left(D_{t|T}^{1+\delta}\varphi_2\right)^{\frac{q}{q-1}}\,dx\,dt\\
&+C\int_0^T\int_{\Omega(B)}\varphi_1^{\frac{\ell(q-p+1-\alpha)-pq}{q-p+1-\alpha}}|\nabla\varphi_1|^{\frac{pq}{q-p+1-\alpha}}\varphi_2^{-\frac{p-1+\alpha}{q-p+1-\alpha}}\left(D_{t|T}^{\delta}\varphi_2\right)^{\frac{q}{q-p+1-\alpha}}\,dx\,dt\\
&+C\int_0^T\int_{\Omega(B)}\varphi_1^{\ell}\varphi_2^{-\frac{\alpha+1}{q-1-\alpha}}\left(D_{t|T}^{1+\delta}\varphi_2\right)^{\frac{q}{q-1-\alpha}}\,dx\,dt\\
&+C\int_0^T\int_{\Omega(B)}\varphi_1^{\frac{\ell(q-(1-\alpha)(p-1))-pq}{q-(1-\alpha)(p-1)}}|\nabla\varphi_1|^{\frac{pq}{q-(1-\alpha)(p-1)}}\varphi_2^{-\frac{(1-\alpha)(p-1)}{q-(1-\alpha)(p-1)}}\left(D_{t|T}^{\delta}\varphi_2\right)^{\frac{q}{q-(1-\alpha)(p-1)}}\,dx\,dt.
\end{aligned}
\tag{4.3.8}
$$

At this stage, we have to distinguish 2 cases.

• Case of $1<q<q_c$: in this case, we take $B=T^{\frac{1}{\theta}}$ where $\theta=p(q-1)/[q-p-\delta p+1+2\delta]$. So, using (1.1.6) and the change of variables : $s=T^{-1}t$, $y=T^{-\frac{1}{\theta}}x$, we get from (4.3.7) that

$$
\int_0^T\int_{\Omega(T^{\frac{1}{\theta}})}|u|^q\tilde{\varphi}\,dx\,dt\leq CT^{-(\delta+1)q'+\frac{n}{\theta}+1},
\tag{4.3.9}
$$

where C is independent of T. Letting $T\to\infty$ in (4.3.9), thanks to $q<q_c$ and the Lebesgue dominated convergence theorem, it yields that

$$
\int_0^\infty\int_{\mathbb{R}^n}|u|^q\,dx\,dt=0,
$$

which implies $u(x,t)=0$ for all t and a.e. x.

• Case of $q=q_c$: let $B=R^{-\frac{1}{\theta}}T^{\frac{1}{\theta}}$, where $1\ll R<T$ is such that T and R do not go simultaneously to ∞. Moreover, from the first case and the fact that $q=q_c$, there exist a positive constant D independent of T such that

$$
\int_0^\infty\int_{\mathbb{R}^n}|u|^q\,dx\,dt\leq D,
$$

which implies that

$$
\int_0^T\int_{\Delta(R^{-\frac{1}{\theta}}T^{\frac{1}{\theta}})}|u|^q\tilde{\varphi}\,dx\,dt\to 0\quad\text{as}\quad T\to\infty,
\tag{4.3.10}
$$

where $\Delta(B):=\{x\in\mathbb{R}^n;\ B<|x|<2B\}$. Repeating a similar calculation as in the subcritical case ($q<q_c$) and using Hölder's inequality instead of Young's one in K_1 and K_3, we get

$$
I_2\leq\frac{1}{3}\int_0^T\int_{\Omega(B)}u^q\tilde{\varphi}\,dx\,dt+C\int_0^T\int_{\Omega(B)}\varphi_1^{\ell}\varphi_2^{-1/(q-1)}\left(D_{t|T}^{1+\delta}\varphi_2\right)^{q'}\,dx\,dt,
$$

$$
K_1\ \leq\ C\left(\int_0^T\int_{\Delta(B)}u^q\tilde{\varphi}\,dx\,dt\right)^{\frac{p-1+\alpha}{q}}
$$

$$\times \left(\int_0^T \int_{\Omega(B)} \varphi_1^{\frac{\ell(q-p+1-\alpha)-pq}{q-p+1-\alpha}} |\nabla \varphi_1|^{\frac{pq}{q-p+1-\alpha}} \varphi_2^{-\frac{p-1+\alpha}{q-p+1-\alpha}} \left(D^{\delta}_{t|T}\varphi_2\right)^{\frac{q}{q-p+1-\alpha}}\,dx\,dt \right)^{\frac{q-p+1-\alpha}{q}},$$

$$K_2 \;\le\; \frac{1}{3}\int_0^T \int_{\Omega(B)} u^q \tilde{\varphi}\,dx\,dt + C\int_0^T \int_{\Omega(B)} \varphi_1^{\ell}\varphi_2^{-\frac{\alpha+1}{q-1-\alpha}} \left(D^{1+\delta}_{t|T}\varphi_2\right)^{\frac{q}{q-1-\alpha}}\,dx\,dt,$$

and

$$\begin{aligned} K_3 \;&\le\; C\left(\int_0^T \int_{\Delta(B)} u^q \tilde{\varphi}\,dx\,dt\right)^{\frac{(1-\alpha)(p-1)}{q}} \\ &\times \left(\int_0^T \int_{\Omega(B)} \varphi_1^{\frac{\ell(q-(1-\alpha)(p-1))-pq}{q-(1-\alpha)(p-1)}} |\nabla \varphi_1|^{\frac{pq}{q-p+1-\alpha}} \varphi_2^{-\frac{(1-\alpha)(p-1)}{q-(1-\alpha)(p-1)}} \left(D^{\delta}_{t|T}\varphi_2\right)^{\frac{q}{q-(1-\alpha)(p-1)}}\,dx\,dt \right)^{\frac{q-(1-\alpha)(p-1)}{q}}. \end{aligned}$$

We conclude that

$$\begin{aligned} &\int_0^T \int_{\Omega(B)} u^q \tilde{\varphi}\,dx\,dt \\ &\le C\int_0^T \int_{\Omega(B)} \varphi_1^{\ell}\varphi_2^{-1/(q-1)} \left(D^{1+\delta}_{t|T}\varphi_2\right)^{\frac{q}{q-1}}\,dx\,dt \\ &+ C\left(\int_0^T \int_{\Delta(B)} u^q \tilde{\varphi}\,dx\,dt\right)^{\frac{p-1+\alpha}{q}} \\ &\times \left(\int_0^T \int_{\Omega(B)} \varphi_1^{\frac{\ell(q-p+1-\alpha)-pq}{q-p+1-\alpha}} |\nabla \varphi_1|^{\frac{pq}{q-p+1-\alpha}} \varphi_2^{-\frac{p-1+\alpha}{q-p+1-\alpha}} \left(D^{\delta}_{t|T}\varphi_2\right)^{\frac{q}{q-p+1-\alpha}}\,dx\,dt \right)^{\frac{q-p+1-\alpha}{q}} \\ &+ C\int_0^T \int_{\Omega(B)} \varphi_1^{\ell}\varphi_2^{-\frac{\alpha+1}{q-1-\alpha}} \left(D^{1+\delta}_{t|T}\varphi_2\right)^{\frac{q}{q-1-\alpha}}\,dx\,dt \\ &+ C\left(\int_0^T \int_{\Delta(B)} u^q \tilde{\varphi}\,dx\,dt\right)^{\frac{(1-\alpha)(p-1)}{q}} \\ &\times \left(\int_0^T \int_{\Omega(B)} \varphi_1^{\frac{\ell(q-(1-\alpha)(p-1))-pq}{q-(1-\alpha)(p-1)}} |\nabla \varphi_1|^{\frac{pq}{q-p+1-\alpha}} \varphi_2^{-\frac{(1-\alpha)(p-1)}{q-(1-\alpha)(p-1)}} \left(D^{\delta}_{t|T}\varphi_2\right)^{\frac{q}{q-(1-\alpha)(p-1)}}\,dx\,dt \right)^{\frac{q-(1-\alpha)(p-1)}{q}}. \end{aligned} \tag{4.3.11}$$

Taking account of $q = q_c$ and the scaled variables $s = T^{-1}t,\ y = R^{\frac{1}{\theta}}T^{-\frac{1}{\theta}}x$, we get

$$\int_0^T \int_{\Omega(B)} |u|^q \tilde{\varphi}\,dx\,dt \le C\,R^{-\frac{n}{\theta}} + CR^{-\frac{n}{\theta}+\frac{pq}{\theta(q-p+1)}} \left(\int_0^T \int_{\Delta(B)} u^q \tilde{\varphi}\,dx\,dt\right)^{\frac{p-1}{q}}$$

Letting $T \to \infty$ and using (4.3.10), we get

$$\int_0^\infty \int_{\mathbb{R}^n} |u|^q\,dx\,dt \le C\,R^{-\frac{n}{\theta}}.$$

Finally, we conclude by letting $R \to \infty$. □

Chapitre 5

Life span of nonnegative solutions to an evolution equation with non local in time nonlinearity

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Abstract

We consider the problem

$$u_t - \Delta|u|^{m-1}u = \int_0^t (t-s)^{-\gamma}|u|^p u(s)\,ds, \quad x \in \mathbb{R}^N,\ t > 0,$$

where $p > 0$, $m > 1$, are real numbers with nonnegative nontrivial continuous bounded initial condition

$$u(x,0) = u_0(x) \not\equiv 0, \quad u_0(x) \geq 0,\ x \in \mathbb{R}^N.$$

We obtain an integral inequality that can be used to find an exponent $p^* \leq p_c$, where p_c is the critical exponent, such that this problem has no global nontrivial solution when $p \leq p^*$. This inequality may also be used to estimate the maximal time $T_{max} > 0$ such that there is a solution for $0 \leq t \leq T_{max}$. This is illustrated with the initial condition $u_\sigma(x,0) = \sigma u_0(x)$, $\sigma > 0$, by obtaining an upper bound of the form $T_{max} \leq C_0\sigma^{-\vartheta}$, for some $\vartheta > 0$.

Keywords : Hyperbolic equation, mild and weak solutions, local existence, Strichartz estimate, blow-up, Riemann-Liouville fractional integrals and derivatives.

5.1 Introduction

In this article, we investigate the maximal interval of existence of the solutions of the problem

$$u_t - \Delta|u|^{m-1}u = \int_0^t (t-s)^{-\gamma}|u|^p u(s)\,ds, \quad x \in \mathbb{R}^N,\ t > 0, \tag{5.1.1}$$

where $p > 0$, $m > 1$, are real numbers with nonnegative nontrivial continuous bounded initial condition

$$u(x,0) = u_0(x) \not\equiv 0, \quad u_0(x) \geq 0, \; x \in \mathbb{R}^N. \tag{5.1.2}$$

If we consider (5.1.1) without nonlinear term, i.e $u_t - \Delta|u|^{m-1}u = 0$, this equation is so-called Porous Medium Equation and have been studied by so many people. Actually many interesting results are obtained so far.

There are a number of physical applications where this model appears in a natural way, mainly to describe processes involving fluid flow, heat transfer or diffusion. Maybe the best known of them is the description of the flow of an isentropic gas through a porous medium, modeled independently by Muskat [56] around 1930. Indeed, this application was at the base of the rigorous mathematical development of the theory. Other applications have been proposed in mathematical biology, spread of viscous fluids, boundary layer theory, and other fields.

When equation (5.1.1) is considered with a nonlinearity of the form $|u|^p u$, it reads

$$u_t - \Delta|u|^{m-1}u = |u|^p u,$$

which is a particular case of (5.1.1) ; it corresponds to $\gamma \to 0$. This equation has been considered by H. J. Kuiper [41]. By life span for initial condition u_0, we mean the least upper bound of all values T such that $[0,T)$ is a maximal interval of existence of a solution. He found that the life span $L(\sigma)$ is bounded by $C\sigma^{-(p+1-m)}$ whenever $u(x,0) = \sigma u_0(x)$, $\sigma > 0$. (see Theorem 3.6 [41]).

Our article is motivated mathematically by the recent papers [10, 47] which deal with the critical exponent and life span for the parabolic equation with nonlocal in time nonlinearity

$$u_t - \Delta u = \int_0^t (t-s)^{\alpha-1}|u|^p u\, ds, \quad x \in \mathbb{R}^N, \; t > 0, \tag{5.1.3}$$

where $u_0 \in C_0(\mathbb{R}^N)$, the space of all continuous functions decaying to zero at infinity, which is a particular case of (5.1.1) ; it corresponds to $m \to 1$.

The main result of this article is to find the exponent

$$p^* = m - 1 + \frac{N\alpha(m-1) + 2\alpha m + 2}{(N-2\alpha)_+}$$

such that if $p \leq p^*$, the problem (5.1.1)-(5.1.2) has no global nontrivial solution.
Suppose u_σ is the solution corresponding to the nontrivial, nonnegative initial condition $u_\sigma(x,0) = \sigma u_0(x)$. Let $[0, T_\sigma)$ be its maximal interval of existence. We obtain a life span of the form

$$L(\sigma) \leq C\sigma^{-\frac{p+1-m}{\alpha+1}}.$$

We will show for equation (5.1.1) a necessary condition for global solution. More precisely, if u is a global solution with initial condition $u(x,0) = u_0(x)$, then an inequality of the form

$$\limsup_{R\to\infty} R^{-S}\int_{B_R} u_0(x)\varphi_1(x/R)\,dx \le C\lambda^{\kappa}, \quad \text{for some } S>0 \text{ and } \kappa>0,$$

must be satisfied. Here φ_1 is the positive eigenfunction corresponding to the principal eigenvalue of the Dirichlet problem on the unit ball B_1, and normalized such that $\int_{B_1}\varphi_1(\xi)\,d\xi = 1$. The constants C and κ depend on N, m, p and α.

The method used to prove the life span and necessary conditions for local and global existence is the test function method [3, 4, 23, 54, 70]. The principle of this method is as follows : we assume, by absurd, that the solution is global, then we use the weak formulation of the equation on $[0,T]$, for some $T>0$ and make an appropriate choice of the test function, then we make a change of variable and finally we take the limit as $T\to\infty$ to get the contradiction.

This paper is organized as follows. In section 5.2, we find the exponent p^*. Finally, we obtain an upper bound for the maximal interval of existence.

5.2 The test function method

Let $t_*>0$. Suppose that u is a solution of (5.1.1)-(5.1.2) on $\mathbb{R}^N\times[0,t_*)$. We assume that

$$1<(1+\alpha)m<p+1.$$

Let λ_R be the principal eigenvalue for the Dirichlet problem on the ball of radius R :

$$-\Delta h(x) = \lambda h(x), \quad x\in B_R,$$

$$h(x)=0, \quad x\in\partial B_R,$$

where $B_R := \{x\in\mathbb{R}^N : |x|<R\}$. We note that $\lambda_R = \lambda_1/R^2$ where λ_1 is the principal eigenvalue of the Laplacian on the unit ball B_1. Let φ_1 denote the unique nonnegative eigenfunction corresponding to the principal eigenvalue λ_1 such that

$$\int_{B_1}\varphi_1(x)dx = 1.$$

Of course φ_1 is radially symmetric : $\varphi_1(x)=\varphi_0(|x|)$.
We define $\varphi_2(t) = \left(1-\dfrac{t}{TR^{\beta}}\right)_+^{\theta}$, where $1<T<R$ is such that when $R\to\infty$ we don't have $T\to\infty$ simultaneously, $\theta\gg 1$ is large enough and

$$\beta = \frac{2(p+1)+(m-1)N}{p+2-m} > 0.$$

We also define

$$\widetilde{\varphi}(x,t) := \varphi_1(x/R)\varphi_2(t) \quad , \quad \varphi(x,t) := D^{\alpha}_{t|TR^{\beta}}\widetilde{\varphi}(x,t)$$

and for $TR^\beta < t_*$,

$$J_R(T) := \int_0^{TR^\beta} \int_{B_R} I^\alpha_{0|t}(u^{p+1})\varphi(x,t)\,dx\,dt.$$

As u is a weak solution, we have

$$\begin{aligned} J_R(T) &= -\int_{B_R} \varphi_1(x/R)u_0(x)D^\alpha_{t|TR^\beta}\varphi_2(0)\,dx \\ &+ \int_0^{TR^\beta}\int_{B_R} uD^{1+\alpha}_{t|TR^\beta}\varphi_2(t)\varphi_1(x/R)\,dx\,dt \\ &+ \int_0^{TR^\beta}\int_{B_R} u^m D^\alpha_{t|TR^\beta}\varphi_2(t)R^{-2}\lambda_1\varphi_1(x/R)\,dx\,dt, \end{aligned}$$

where we have used the fact that $\Delta\varphi_1(x/R) = R^{-2}\lambda_1\varphi_1(x/R)$.
Using (1.1.7) and letting $V_R := \displaystyle\int_{B_R} \varphi_1(x/R)u_0(x)\,dx$, we get

$$\begin{aligned} & J_R(T) + C\,T^{-\alpha}R^{-\alpha\beta}V_R \\ &= \int_0^{TR^\beta}\int_{B_R} uD^{1+\alpha}_{t|TR^\beta}\varphi_2(t)\varphi_1(x/R)\,dx\,dt + \int_0^{TR^\beta}\int_{B_R} u^m D^\alpha_{t|TR^\beta}\varphi_2(t)R^{-2}\lambda_1\varphi_1(x/R)\,dx\,dt \\ &= \int_0^{TR^\beta}\int_{B_R} [u^{p+1}\varphi_1(x/R)\varphi_2(t)]^{\frac{1}{p+1}}\varphi_1(x/R)^{\frac{p}{p+1}}\varphi_2(t)^{-\frac{1}{p+1}}D^{1+\alpha}_{t|TR^\beta}\varphi_2(t)\,dx\,dt \\ &\quad + \lambda_1 R^{-2}\int_0^{TR^\beta}\int_{B_R} [u^{p+1}\varphi_1(x/R)\varphi_2(t)]^{\frac{m}{p+1}}\varphi_1(x/R)^{\frac{p+1-m}{p+1}}\varphi_2(t)^{-\frac{m}{p+1}}D^\alpha_{t|TR^\beta}\varphi_2(t)\,dx\,dt \\ &\le J_R(T)^{\frac{1}{p+1}}\left[\int_0^{TR^\beta}\int_{B_R}\varphi_1(x/R)\varphi_2(t)^{-\frac{1}{p}}(D^{1+\alpha}_{t|TR^\beta}\varphi_2(t))^{\frac{p+1}{p}}\,dx\,dt\right]^{\frac{p}{p+1}} \\ &\quad + \lambda_1 R^{-2}J_R(T)^{\frac{m}{p+1}}\left[\int_0^{TR^\beta}\int_{B_R}\varphi_1(x/R)\varphi_2(t)^{-\frac{m}{p+1-m}}(D^\alpha_{t|TR^\beta}\varphi_2(t))^{\frac{p+1}{p+1-m}}\,dx\,dt\right]^{\frac{p+1-m}{p+1}}. \end{aligned}$$

We multiply by T^α and use (1.1.6); we obtain

$$\begin{aligned} H_R(T) + R^{-\alpha\beta}V_R \;&\le\; H_R(T)^{\frac{1}{p+1}}R^{-\beta(1+\alpha)}\left[\int_0^{TR^\beta}\int_{B_R} T^{-\frac{p+1+\alpha}{p}}\varphi_1(x/R)\varphi_2(t)^{\delta_1}\,dx\,dt\right]^{\frac{p}{p+1}} \\ &+ H_R(T)^{\frac{m}{p+1}}\lambda_1 R^{-(2+\alpha\beta)}\left[\int_0^{TR^\beta}\int_{B_R} T^{-\frac{\alpha m}{p+1-m}}\varphi_1(x/R)\varphi_2(t)^{\delta_2}\,dx\,dt\right]^{\frac{p+1-m}{p+1}}, \end{aligned}$$

where

$$H_R(T) = T^\alpha J_R(T), \quad \delta_1 = \frac{p(\theta-1)-\alpha(p+1)-1}{p\theta} \quad \text{and} \quad \delta_2 = \frac{(p+1)(\theta-\alpha)-m\theta}{\theta(p+1-m)}.$$

Making the change of variable $\xi = x/R$ and $\tau = t/R^\beta$, we deduce that

$$H_R(T) + R^{-\alpha\beta}V_R \le H_R(T)^{\frac{1}{p+1}}R^{s_1}A(T)^{\frac{p}{p+1}} + H_R(T)^{\frac{m}{p+1}}\lambda_1 R^{s_2}B(T)^{\frac{p+1-m}{p+1}}, \tag{5.2.1}$$

where

$$s_1 = -\beta(1+\alpha) + \frac{p(\beta+N)}{p+1}, \qquad s_2 = -2 + N + \beta(1-\alpha) - \frac{(\beta+N)m}{p+1},$$

$$A(T) = T^{-\frac{p+1+\alpha}{p}} \int_0^T \int_{B_1} \varphi_1(\xi)\varphi_2(\tau)^{\delta_1}\, d\xi\, d\tau,$$

and

$$B(T) = T^{-\frac{\alpha m}{p+1-m}} \int_0^T \int_{B_1} \varphi_1(\xi)\varphi_2(\tau)^{\delta_2}\, d\xi\, d\tau.$$

As

$$\beta = \frac{2(p+1)+(m-1)N}{p+2-m} > 0, \tag{5.2.2}$$

we have

$$s = s_1 = s_2 = \frac{(p+1-m)N - \alpha[2(p+1)+N(m-1)] - 2}{p+2-m}. \tag{5.2.3}$$

It is our aim to use (5.2.1) to obtain information on the relationship between the initial condition and the length of the maximum interval of existence.

Theorem 5.2.1 *If $s \le 0$, that is to say*

$$p \le p^* = m - 1 + \frac{N\alpha(m-1) + 2\alpha m + 2}{(N-2\alpha)_+},$$

then problem (5.1.1)-(5.1.2) has no global nontrivial solution.

Proof. In the case of $s < 0$ (i.e $p < p^*$), we use (1.1.3), (1.1.5), and (5.2.1) to obtain

$$\lim_{R\to\infty} \int_0^{TR^\beta} \int_{B_R} I^\alpha_{0|t}(u^{p+1})\varphi(x,t)\, dx\, dt = \lim_{R\to\infty} \int_0^{TR^\beta} \int_{B_R} u^{p+1}\widetilde{\varphi}(x,t)\, dx\, dt = 0,$$

i.e.

$$\int_0^\infty \int_{\mathbb{R}^N} u^{p+1}\, dx\, dt = 0, \tag{5.2.4}$$

which implies that $u \equiv 0$ is the only global solution.
If $s = 0$ (i.e $p = p^*$), we note that $J_R(T)$ is uniformly bounded for all R. Therefore by taking the limit as $R \to \infty$ and then $T \to \infty$ on the right-hand side of (5.2.1), we obtain again (5.2.4). □

5.3 Life Span of a solution

Suppressing argument and subscripts (5.2.1) becomes

$$H + R^{-\alpha\beta} V \le H^{\frac{1}{p+1}} R^s A^{\frac{1}{p+1}} + H^{\frac{m}{p+1}} \lambda R^s B^{\frac{p+1-m}{p+1}}. \tag{5.3.1}$$

We will use this to obtain an estimate for V. First we state some lemmas.

Lemma 5.3.1 *Suppose that $a, b, r,$ and q are positive constants. Define the functions $F(x) := ax^q - bx^r, G(x) := ax^{-q} + bx^r$ on $0 < x < \infty$. Then*

$$\max_{x>0} F(x) = \left(1 - \frac{q}{r}\right) a^{\frac{r}{r-q}} \left(\frac{q}{br}\right)^{\frac{q}{r-q}},$$

$$\min_{x>0} G(x) = \left(1 + \frac{q}{r}\right) a^{\frac{r}{r+q}} \left(\frac{br}{q}\right)^{\frac{q}{r+q}}.$$

Lemma 5.3.2 *Let $0 < w_1, w_2 < 1, w_1 \neq w_2$. On $[0, \infty)$ define*

$$\Upsilon(x) := max(x^{w_1}, x^{w_2}).$$

Let η be an arbitrary positive number, then

$$\Psi(w_1, w_2; \eta) := \max_x (\eta \Upsilon(x) - x) = \max_i \left((1 - w_i) w_i^{\frac{w_i}{1-w_i}} \eta^{\frac{1}{1-w_i}}\right).$$

For η sufficiently large

$$\Psi(w_1, w_2; \eta) = (1 - \overline{w}) \overline{w}^{\frac{\overline{w}}{1-\overline{w}}} \eta^{\frac{1}{1-\overline{w}}}, \tag{5.3.2}$$

where $\overline{w} = max(w_1, w_2)$.

We will use the notation

$$J_m := \left(1 - \frac{m}{p+1}\right) \left(\frac{m}{p+1}\right)^{\frac{m}{p+1-m}}.$$

Then, for η sufficiently large,

$$\Psi\left(\frac{1}{p+1}, \frac{m}{p+1}, \eta\right) = J_m \eta^{\frac{p+1}{p+1-m}}.$$

Theorem 5.3.3 *If u is a nonnegative solution of (5.1.1)-(5.1.2) on $B_{R_*} \times [0, t_*)$ and s given by (5.2.3), then for all $(R, T) \in \left\{(\rho, \tau) : R_0 \leq \rho \leq R_*, 0 \leq \tau \leq t_* \rho^{-\beta}\right\}$, we have*

$$R^{-\alpha\beta} \int_{B_R} u_0(x) \varphi_1(x/R)\, dx \leq \Psi\left(\frac{1}{p+1}, \frac{m}{p+1}; \left[A(T)^{\frac{p}{p+1}} + \lambda B(T)^{\frac{p+1-m}{p+1}}\right] R^s\right). \tag{5.3.3}$$

In particular, if u is a global nonnegative solution then

$$\limsup_{R\to\infty} R^{-S} \int_{B_R} u_0(x) \varphi_1(x/R)\, dx \leq J_{\overline{m}} \inf_T \left[A(T)^{\frac{p}{p+1}} + \lambda B(T)^{\frac{p+1-m}{p+1}}\right]^{\frac{p+1}{p+1-m}}, \tag{5.3.4}$$

where

$$S = \frac{s(p+1)}{p+1-m} + \alpha\beta.$$

Proof. For the sake of convenience we define

$$\Theta(T) = A(T)^{\frac{p}{p+1}} + \lambda B(T)^{\frac{p+1-m}{p+1}}.$$

From (5.3.1) we see that

$$R^{-\alpha\beta} V \leq \Upsilon(J)\Theta(T)R^s - J, \tag{5.3.5}$$

where

$$\Upsilon(\mu) := max\left\{\mu^{\frac{1}{p+1}}, \mu^{\frac{m}{p+1}}\right\}.$$

Then by Lemma 5.2, we have (5.3.3). For R sufficiently large we can use equation (5.3.2) to conclude the validity of (5.3.4). □

Corollary 5.3.4 *Suppose that u is a nonnegative global solution. Then*

$$\limsup_{R\to\infty} R^{-S} \int_{B_R} u_0(x)\varphi_1(x/R)\,dx \leq J_{\overline{m}} K_m^{\frac{p+1}{p+1-\overline{m}}} \lambda^{\frac{(p+1)(\alpha+1)}{(p+1-m)(p+1-(\alpha+1)(m-1))}},$$

where K_m is a constant depending on m, θ and p.

Proof. We easily obtain

$$A(T) \leq A_0 \equiv \frac{T^{-\frac{1+\alpha}{p}} p}{p\theta - \alpha(p+1) - 1},$$

and

$$B(T) \leq B_0 \equiv \frac{(p+1-m)T^{\frac{(p+1)-m(1+\alpha)}{p+1-m}}}{\theta(p+1-m) + (p+1)(1-\alpha) - m}.$$

Then, from (5.3.5), we have

$$R^{-\alpha\beta} \int_{B_R} u_0(x)\varphi_1(x/R)\,dx \leq R^s \Theta_0(T)\Upsilon(J) - J,$$

where

$$\Theta(T) \leq \Theta_0(T) := \alpha_0 T^{-\frac{(\alpha+1)}{p+1}} + \beta_0 T^{\frac{p+1-m(1+\alpha)}{p+1}},$$

with

$$\alpha_0 := \left[\frac{p}{\theta p - \alpha(p+1) - 1}\right]^{\frac{p}{p+1}}, \qquad \beta_0 := \lambda\left[\frac{p+1-m}{(p+1-m)\theta + (p+1)(1-\alpha) - m}\right]^{\frac{p+1-m}{p+1}}.$$

By Lemma 5.1, we get

$$\begin{aligned} \underline{\Theta}_0 &:= \min \Theta_0(T) \\ &= \left(1 + \frac{\alpha+1}{p+1-m(1+\alpha)}\right) \alpha_0^{\frac{p+1-m(1+\alpha)}{p+1-(\alpha+1)(m-1)}} \left[\beta_0 \frac{(p+1) - m(1-\alpha)}{\alpha+1}\right]^{\frac{\alpha+1}{p+1-(\alpha+1)(m-1)}} \end{aligned}$$

$$
\begin{aligned}
= & \left[\frac{p+1-(\alpha+1)(m-1)}{(p+1)-m(1+\alpha)}\right]\left[\frac{p+1-m(\alpha+1)}{\alpha+1}\right]^{\frac{\alpha+1}{p+1-(\alpha+1)(m-1)}} \\
& \times\lambda^{\frac{\alpha+1}{p+1-(\alpha+1)(m-1)}}\left[\frac{p}{\theta p-\alpha(p+1)-1}\right]^{\frac{p[p+1-m(1+\alpha)]}{(p+1)[p+1-(\alpha+1)(m-1)]}} \\
& \times\left[\frac{p+1-m}{\theta(p+1-m)+(p+1)(1-\alpha)-m}\right]^{\frac{(p+1-m)(\alpha+1)}{(p+1)[p+1-(\alpha+1)(m-1)]}}.
\end{aligned}
$$

As $\theta \gg 1$ we have $\underline{\Theta}_0 \leq K_m \lambda^{\frac{\alpha+1}{p+1-(m-1)(\alpha+1)}}$. Then after substituting this into equation (5.3.4), the proof is complete. $\square$

When we are dealing with the problem originally considered by Fujita ($m \to 1$ and $\alpha \to 0$), then $J_1 = p(p+1)^{-\frac{(p+1)}{p}}$ and $K_1 = (p+1)p^{-\frac{p}{p+1}}$, and we see that the above inequality reduces to

$$
\limsup_{R\to\infty} R^{-N+2/p}\int_{B_R} u_0(x)\varphi_1(x/R)\,dx \leq C\lambda^{1/p}.
$$

This is precisely the result found in [45]. Note that if $m > 1$ and $\alpha \to 0$, we recover the case of Kuiper [41]. As it is done in that article, we can deduce the following result.

Corollary 5.3.5 *When $N \geq S$, Theorem 5.1 and Corollary 5.4 remain valid if we replace*

$$
\limsup_{R\to\infty} R^{-S}\int_{B_R} u_0(x)\varphi_1(x/R)\,dx \qquad by \qquad \liminf_{|x|\to\infty}(|x|^{N-S}u_0(x)).
$$

Proof. The statement of this corollary follows from the inequalities :

$$
\begin{aligned}
\lim_{R\to\infty} R^{-S}V & \geq \lim_{R\to\infty} R^{-S}\int_{B_R\setminus B_k}\inf_{R\geq|x|\geq k}(|x|^{N-S}u_0(x))R^{S-N}\varphi_1(x/R)\,dx \\
& \geq \lim_{R\to\infty}\inf_{R\geq|x|\geq k}(|x|^{N-S}u_0(x))\int_{B_R\setminus B_k} R^{-N}\varphi_1(x/R)\,dx \\
& = \lim_{R\to\infty}\inf_{R\geq|x|\geq k}(|x|^{N-S}u_0(x))\int_{B_1\setminus B_{k/R}}\varphi_1(\zeta)\,d\zeta \\
& = \inf_{|x|\geq k}(|x|^{N-S}u_0(x)).
\end{aligned}
$$

The proof is complete by letting $k \to \infty$. $\square$

Now, we consider the problem (5.1.1)-(5.1.2). By the life span for initial condition u_0, we mean the least upper bound of all values T such that $[0, T)$ is the maximal interval of existence of the solution to (5.1.1)-(5.1.2). Let us fix $u_0 \not\equiv 0$ and $u_0 \geq 0$ for all $x \in \mathbb{R}^N$. We denote by $L(\sigma)$, $\sigma > 0$, the life span corresponding to initial condition σu_0. There exists a value Λ such that

$$
\Lambda R^{-\alpha\beta}V = \Psi(R^s\Theta(T_M)),
$$

where T_M is the value of T at which $\Theta(T)$ attains its minimum value.

Let Θ_L denote the restriction of Θ to the interval $[0, T_M)$. If we take $\sigma > \Lambda$, then $L(\sigma) < \infty$, and we have

$$L(\sigma) \leq \Theta_L^{-1}(R^{-s}\Psi^{-1}(\sigma R^{\alpha\beta} V)). \tag{5.3.6}$$

In the next result we use this inequality to obtain an explicit upper bound for the life span of a solution.

Theorem 5.3.6 *Let u_0 be a nonnegative nontrivial continuous function in $\mathbb{R}^N$. Then, there exists positive constants Λ_m, C and σ_1 so that the life span $L(\sigma)$ corresponding to the initial condition σu_0 with $\sigma > \Lambda_m$ satisfies*

$$L(\sigma) \leq C\sigma^{-\frac{p+1-m}{\alpha+1}}. \tag{5.3.7}$$

Proof. Decreasing the value of T_M to a value T_m if needed, we may assume that the function Θ_0, introduced above, is decreasing on $(0, T_M)$. We can choose Λ_m such that in the meantime $\Lambda_m R^{-\alpha\beta} V \geq \Psi(R^s \Theta(T_m))$ and $\Lambda_m R^{-\alpha\beta} V \geq C_1$, where C_1 is a sufficiently large constant so that whenever $\sigma > \Lambda_m$, then

$$\Psi^{-1}(\sigma R^{-\alpha\beta} V) = [(1-\overline{w})^{-1}\overline{w}^{-\frac{\overline{w}}{1-\overline{w}}}]^{1-\overline{w}}(\sigma R^{-\alpha\beta}V)^{\frac{p+1-m}{p+1}},$$

with $\overline{w} = m/(p+1)$. We write

$$\Psi^{-1}(\sigma R^{-\alpha\beta} V) = \gamma_0 R^{-\alpha\beta\frac{p+1-m}{p+1}} V^{\frac{p+1-m}{p+1}} \sigma^{\frac{p+1-m}{p+1}},$$

where $\gamma_0 = (p+1)(p+1-m)^{-\frac{p+1-m}{p+1}} m^{-\frac{m}{p+1}}$. Since

$$\Theta(T) \leq \Theta_0(T) \leq \alpha_0 T^{-\frac{\alpha+1}{p+1}} + \beta_0 T_m^{\frac{p+1-m(1+\alpha)}{p+1}},$$

on $[0, T_m)$, it follows that

$$\Theta_L^{-1} \leq \left[\frac{\eta - \beta_0 T_m^{\frac{p+1-m(1+\alpha)}{p+1}}}{\alpha_0}\right]^{-\frac{p+1}{\alpha+1}}, \quad \text{for} \quad \eta > \beta_0 T_m^{\frac{p+1-m(1+\alpha)}{p+1}}.$$

Let $[0, T_{\max})$ be the maximal interval of existence of u and $T \in [0, T_{\max})$. We define

$$G(R,\sigma) := \alpha_0^{p+1}[\gamma_0 R^{-s} R^{-\alpha\beta} V^{\frac{p+1-m}{p+1}} \sigma^{\frac{p+1-m}{p+1}} - \delta_0]^{-\frac{p+1}{\alpha+1}},$$

where $\delta_0 = \beta_0 T_m^{\frac{p+1-m(1+\alpha)}{p+1}}$. Whenever $T < L(\sigma)$ we have $T \leq G(R,\sigma)$. Therefore

$$L(\sigma) \leq G(R,\sigma). \tag{5.3.8}$$

It is easily seen that this implies equation (5.3.7).

$\square$

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